On a hyperbolic coefficient inverse problem via partial dynamic boundary measurements

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Abstract

This paper is devoted to the identification of the unknown smooth coefficient $c$ entering the hyperbolic equation $c(x)\partial^2_t u - \Delta u = 0$ in a bounded smooth domain in $\mathbb{R}^d$ from partial (on part of the boundary) dynamic boundary measurements. In this paper we prove that the knowledge of the partial Cauchy data for this class of hyperbolic PDE on any open subset $\Gamma$ of the boundary determines explicitly the coefficient $c$ provided that $c$ is known outside a bounded domain. Then, through construction of appropriate test functions by a geometrical control method, we derive a formula for calculating the coefficient $c$ from the knowledge of the difference between the local Dirichlet to Neumann maps.

Key words. inverse problem, hyperbolic equation, geometric control, identification

1 Introduction

In this paper, we present a differently method for multidimensional Coefficient Inverse Problems (CIPs) for a class of hyperbolic Partial Differential Equations (PDEs). In the literature, the reader can find many key investigations in this kind of inverse problems, see, e.g. [2, 4, 6, 7, 16, 17, 22, 27, 28, 33, 34] and references cited there. L. Beilina and M.V. Klibanov have deeply studied this important problem in various recently works [4, 5]. In [4], the authors have introduced a new globally convergent numerical method to solve a coefficient inverse problem associated to a hyperbolic PDE. The development of globally convergent numerical methods for multidimensional CIPs has started, as a first generation, from the developments found in [18, 19, 20]. Else, A. G. Ramm and Rakesh have developed a general method for proving uniqueness theorems for multidimensional inverse problems. For the two dimensional case, Nachman [22] proved an uniqueness result for CIPs for some elliptic equation. Moreover, we find the works of L. Päivärinta and V. Serov [23, 29] about the same issue, but for elliptic equations.

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In other manner, the author Y. Chen has treated in [12] the Fourier transform of the hyperbolic equation similar to ours with the unknown coefficient $c(x)$. Unlike this, we derive, using as weights particular background solutions constructed by a geometrical control method, asymptotic formulas in terms of the partial dynamic boundary measurements (Dirichlet-to-Neumann map) that are caused by the small perturbations. These asymptotic formulae yield the inverse Fourier transform of unknown coefficient.

The ultimate objective of the work described in this paper is to determine, effectively, the unknown smooth coefficient $c$ entering a class of hyperbolic equation in a bounded smooth domain in $\mathbb{R}^d$ from partial (on part of the boundary) dynamic boundary measurements. The main difficulty appears in boundary measurements, is that the formulation of our boundary value problem involves unknown boundary values. This problem is well known in the study of the classical elliptic equations, where the characterization of the unknown Neumann boundary value in terms of the given Dirichlet datum is known as the Dirichlet-to-Neumann map. But, the problem of determining the unknown boundary values also occurs in the study of hyperbolic equations formulated in a bounded domain.

As our main result we develop, using as weights particular background solutions constructed by a geometrical control method, asymptotic formulas for appropriate averaging of the partial dynamic boundary measurements that are caused by the small perturbations of coefficient according to a parameter $\alpha$. Assume that the coefficient is known outside a bounded domain $\Omega$, and suppose that we know explicitly the value of $\lim_{\alpha \to 0^+} c(x)$ for $x \in \Omega$. Then, the developed asymptotic formulae yield the inverse Fourier transform of the unknown part of this coefficient.

In the subject of small volume perturbations from a known background material associated to the full time-dependent Maxwell’s equations, we have derived asymptotic formulas to identify their locations and certain properties of their shapes from dynamic boundary measurements [13]. The present paper represents a different investigation of this line of work.

As closely related stationary identification problems we refer the reader to [11, 15, 22, 30] and references cited there.

## 2 Problem formulation

Let $\Omega$ be a bounded, smooth subdomain of $\mathbb{R}^d$ with $d \leq 3$, (the assumption $d \leq 3$ is necessary in order to obtain the appropriate regularity for the solution using classical Sobolev embedding, see Brezis [9]). For simplicity we take $\partial \Omega$ to be $C^\infty$, but this condition could be considerably weakened. Let $n = n(x)$ denote the outward unit normal vector to $\Omega$ at a point on $\partial \Omega$. Let $T > 0$, $x_0 \in \mathbb{R}^d \setminus \Omega$ and let $\Omega'$ be a smooth subdomain of $\Omega$. We denote by $\Gamma \subset \subset \partial \Omega$ as a measurable smooth open part of the boundary $\partial \Omega$. Throughout this paper we shall use quite standard $L^2$–based Sobolev spaces to measure regularity.

As the forward problem, we consider the Cauchy problem for a hyperbolic PDE

$$c(x)v_{tt} - \Delta v = 0 \quad \text{in } \mathbb{R}^d \times (0, T)$$

$$v(x, 0) = 0, \quad v_t(x, 0) = \delta(x - x_0) + \chi(\Omega)\psi, \quad (2)$$
where $\chi(\Omega)$ is the characteristic function of $\Omega$ and $\psi \in C^\infty(\mathbb{R}^d)$ that $\psi(x) \neq 0$, $\forall x \in \overline{\Omega}$.

Equation (1) governs a wide range of applications, including e.g., propagation of acoustic and electromagnetic waves.

We assume that the coefficient $c(x)$ of equation (1) is such that

$$c(x) = \begin{cases} 
  c_\alpha(x) = c_0(x) + \alpha c_1(x) & \text{for } x \in \Omega, \\
  c_2(x) = \text{const.} > 0 & \text{for } x \in \mathbb{R}^d \setminus \Omega;
\end{cases}$$

where $c_i(x) \in C^2(\overline{\Omega})$ for $i = 0, 1$ with $c_1 \equiv 0$ in $\Omega \setminus \overline{\Omega}'$, and $M := \sup\{c_1(x); x \in \Omega\}'$, (4)

where $\Omega'$ is a smooth subdomain of $\Omega$ and $M$ is a positive constant. We also assume that $\alpha > 0$, the order of magnitude of the small perturbations of coefficient, is sufficiently small that

$$|c_\alpha(x)| \geq c_* > 0, x \in \overline{\Omega},$$

where $c_*$ is a positive constant.

Suppose that the positive number $c_2$ is given. In this paper we assume that the function $c(x)$ is unknown in the domain $\Omega$. Our purpose is the determination of $c(x)$ for $x \in \Omega$, assuming that the following function $g(x,t)$ is known for the single source position $x_0 \in \mathbb{R}^d \setminus \overline{\Omega}$. Therefore, as done for the Dirichlet boundary conditions in [5], we set the Neumann boundary conditions:

$$\frac{\partial v}{\partial n}|_{\partial \Omega \times (0,T)} = g(x,t).$$

The knowledge of $c(x)$ outside of $\Omega$ ($c(x) = c_2$ in $\mathbb{R}^d \setminus \Omega$), and the boundary function $g(x,t)$ allow us to determine uniquely the function $v(x,t)$ for $x \in \mathbb{R}^d \setminus \Omega$ as solution of the boundary value problem for equations (1)-(2) with initial conditions in (2) and with the boundary conditions (6). Therefore, one can uniquely determine the function $f(x,t) = v|_{\partial \Omega \times (0,T)}$.

Then, we can now consider an initial boundary value problem only in the domain $\Omega \times (0,T)$. Thus, the function $v$ satisfying (1)-(2) is en particular solution of the following initial boundary value problem

$$\begin{cases} 
  (c_\alpha \partial_t^2 - \Delta)u_\alpha = 0 & \text{in } \Omega \times (0,T), \\
  u_\alpha|_{t=0} = \varphi, \partial_t u_\alpha|_{t=0} = \psi & \text{in } \Omega, \\
  u_\alpha|_{\partial \Omega \times (0,T)} = f.
\end{cases}$$

Define $u$ to be the solution of the hyperbolic equation in the homogeneous situation ($\alpha = 0$). Thus, $u$ satisfies

$$\begin{cases} 
  (c_0 \partial_t^2 - \Delta)u = 0 & \text{in } \Omega \times (0,T), \\
  u|_{t=0} = \varphi, \partial_t u|_{t=0} = \psi & \text{in } \Omega, \\
  u|_{\partial \Omega \times (0,T)} = f.
\end{cases}$$
Here \( \varphi \in C^\infty(\Omega) \) and \( f \in C^\infty(0, T; C^\infty(\partial \Omega)) \) are subject to the compatibility conditions

\[
\partial_t^{2l} f \big|_{t=0} = (\Delta^l \varphi) \big|_{\partial \Omega} \quad \text{and} \quad \partial_t^{2l+1} f \big|_{t=0} = (\Delta^l \psi) \big|_{\partial \Omega}, \quad l = 1, 2, \ldots
\]

which give that (8) has a unique solution in \( C^\infty([0, T] \times \Omega) \), see [14]. It is also well-known that (7) has a unique weak solution \( u_\alpha \in C^\infty(0, T; H^1(\Omega)) \cap C^1(0, T; L^2(\Omega)) \), see [21], [14]. Indeed, from [21] we have that \( \frac{\partial u_\alpha}{\partial n} \big|_{\partial \Omega} \) belongs to \( L^2(0, T; L^2(\partial \Omega)) \).

Now, we define \( \Gamma_c := \partial \Omega \setminus \Gamma \), and we introduce the trace space

\[
\tilde{H}^{\frac{1}{2}}(\Gamma) = \left\{ v \in H^{\frac{1}{2}}(\partial \Omega \times (0, T)), v \equiv 0 \text{ on } \Gamma_c \times (0, T) \right\}.
\]

It is known that the dual of \( \tilde{H}^{\frac{1}{2}}(\Gamma) \) is \( H^{-\frac{1}{2}}(\Gamma) \).

To introduce the local Dirichlet to Neumann map associated to our problem, we firstly define the function \( \tilde{f} = \chi(\Gamma)f \) for \((x, t) \in \partial \Omega \times (0, T)\), where \( \chi(\Gamma) \) is the characteristic function of \( \Gamma \). Then, we have

\[
\tilde{f} = f \big|_{\Gamma \times (0, T)} \quad \text{and} \quad \tilde{f} \in \tilde{H}^{\frac{1}{2}}(\Gamma).
\]

Therefore, we define the local Dirichlet to Neumann map associated to coefficient \( c_\alpha \) by:

\[
\Lambda_\alpha(\tilde{f}) = \frac{\partial u_\alpha}{\partial n} \big|_{\Gamma} \text{ for } \tilde{f} \in \tilde{H}^{\frac{1}{2}}(\Gamma),
\]

where \( u_\alpha \) is the solution of (7). Let \( u \) denote the solution to the hyperbolic equation (8) with the Dirichlet boundary condition \( u = f \) on \( \partial \Omega \times (0, T) \). Then, the local Dirichlet to Neumann map associated to \( c_0 \) is \( \Lambda_0(\tilde{f}) = \frac{\partial u}{\partial n} \big|_{\Gamma} \) for \( \tilde{f} \in \tilde{H}^{\frac{1}{2}}(\Gamma) \).

Our problem can be stated as follows:

**Inverse problem.** Suppose that the smooth coefficient \( c(x) \) satisfies (3)-(4)-(5), where the positive number \( c_2 \) is given. Assume that the function \( c(x) \) is unknown in the domain \( \Omega \) and \( \tilde{f} \) is given by (9). Is it possible to determine the coefficient \( c_\alpha(x) \) from the knowledge of the difference between the local Dirichlet to Neumann maps \( \Lambda_\alpha - \Lambda_0 \) on \( \Gamma \), if we know explicitly the value of \( \lim_{\alpha \to 0^+} c_\alpha(x) \) for \( x \in \Omega \)?

To give a positive answer, we will develop an asymptotic expansions of an "appropriate averaging" of \( \frac{\partial u_\alpha}{\partial n} \) on \( \Gamma \times (0, T) \), using particular background solutions as weights. These particular solutions are constructed by a control method as it has been done in the original work [33] (see also [8], [10], [24], [25] and [34]). It has been known for some time that the full knowledge of the (hyperbolic) Dirichlet to Neumann map \((u_\alpha|_{\partial \Omega \times (0, T)} \mapsto \frac{\partial u_\alpha}{\partial n}|_{\partial \Omega \times (0, T)})\) uniquely determines conductivity, see [26], [31]. Our identification procedure can be regarded as an important attempt to generalize the results of [26] and [31] in the case of partial knowledge (i.e., on only part of the boundary) of the Dirichlet to Neumann map to determine the coefficient of the hyperbolic equation considered above. The question of uniqueness of this inverse problem can be addressed positively via the method of Carleman estimates, see, e.g., [17, 19].
3 The Identification Procedure

Before describing our identification procedure, let us introduce the following cutoff function $\beta(x) \in C^\infty_0(\Omega)$ such that $\beta \equiv 1$ on $\Omega'$ and let $\eta \in \mathbb{R}^d$.

We will take in what follows $\varphi(x) = e^{i\eta \cdot x}$, $\psi(x) = -i|\eta|e^{i\eta \cdot x}$, and $f(x, t) = e^{i\eta \cdot x - i|\eta|t}$ and assume that we are in possession of the boundary measurements of $\frac{\partial u_\alpha}{\partial n}$ on $\Gamma \times (0, T)$.

This particular choice of data $\varphi, \psi, f$ implies that the background solution $u$ of the wave equation (8) in the homogeneous background medium can be given explicitly.

Suppose now that $T$ and the part $\Gamma$ of the boundary $\partial \Omega$ are such that they geometrically control $\Omega$ which roughly means that every geometrical optic ray, starting at any point $x \in \Omega$ at time $t = 0$ hits $\Gamma$ before time $T$ at a non diffractive point, see [3]. It follows from [32] (see also [1]) that there exists (a unique) $g_\eta \in H^1_0(0, T; TL^2(\Gamma))$ (constructed by the Hilbert Uniqueness Method) such that the unique weak solution $w_\eta$ to the wave equation

\[
\begin{cases}
(c_0 \partial_t^2 - \Delta)w_\eta = 0 & \text{in } \Omega \times (0, T), \\
w_\eta|_{t=0} = \beta(x)e^{i\eta \cdot x} \in H^1_0(\Omega), \\
\partial_t w_\eta|_{t=0} = 0 & \text{in } \Omega, \\
w_\eta|_{\Gamma \times (0, T)} = g_\eta, \\
w_\eta|_{\partial \Omega \setminus \Gamma \times (0, T)} = 0,
\end{cases}
\]

satisfies $w_\eta(T) = \partial_t w_\eta(T) = 0$.

Let $\theta_\eta \in H^1(0, T; L^2(\Gamma))$ denote the unique solution of the Volterra equation of second kind

\[
\begin{cases}
\partial_t \theta_\eta(x, t) + \int_0^T e^{-i|\eta|(s-t)}(\theta_\eta(x, s) - i|\eta|\partial_t \theta_\eta(x, s)) \, ds = g_\eta(x, t) & \text{for } x \in \Gamma, t \in (0, T), \\
\theta_\eta(x, 0) = 0 & \text{for } x \in \Gamma.
\end{cases}
\]

We can refer to the work of Yamamoto in [34] who conceived the idea of using such Volterra equation to apply the geometrical control for solving inverse source problems.

The existence and uniqueness of this $\theta_\eta$ in $H^1(0, T; L^2(\Gamma))$ for any $\eta \in \mathbb{R}^d$ can be established using the resolvent kernel. However, observing from differentiation of (11) with respect to $t$ that $\theta_\eta$ is the unique solution of the ODE:

\[
\begin{cases}
\partial_t^2 \theta_\eta - \theta_\eta = e^{i|\eta|t}\partial_t(e^{-i|\eta|t}g_\eta) & \text{for } x \in \Gamma, t \in (0, T), \\
\theta_\eta(x, 0) = 0, \partial_t \theta_\eta(x, T) = 0 & \text{for } x \in \Gamma,
\end{cases}
\]

the function $\theta_\eta$ may be find (in practice) explicitly with variation of parameters and it also immediately follows from this observation that $\theta_\eta$ belongs to $H^2(0, T; L^2(\Gamma))$.

We introduce $v_\eta$ as the unique weak solution (obtained by transposition) in $C^0(0, T; L^2(\Omega)) \cap$
\( C^1(0, T; H^{-1}(\Omega)) \) to the wave equation

\[
\begin{cases}
(c_0 \partial_t^2 - \Delta) v_\eta = 0 \quad \text{in } \Omega \times (0, T), \\
v_{\eta}|_{t=0} = 0 \quad \text{in } \Omega, \\
\partial_t v_{\alpha, \eta}|_{t=0} = i \nabla \cdot (\eta c_1(x)e^{i\eta \cdot x}) \in L^2(\Omega), \\
v_\eta|_{\partial \Omega \times (0, T)} = 0.
\end{cases}
\]

Then, the following holds.

**Proposition 3.1** Suppose that \( \Gamma \) and \( T \) geometrically control \( \Omega \). For any \( \eta \in \mathbb{R}^d \) we have

\[
\int_0^T \int_\Gamma g_\eta \Lambda_0(v_\eta) \, d\sigma(x) \, dt = |\eta|^2 \int_{\Omega'} c_1(x)e^{2i\eta \cdot x} \, dx.
\]

Here \( d\sigma(x) \) means an elementary surface for \( x \in \Gamma \).

**Proof.** Let \( v_\eta \) be the solution of (13). From [21] [Theorem 4.1, page 44] it follows that \( \Lambda_0(v_\eta) = \partial v_\eta/\partial n|_\Gamma \in L^2(0, T; L^2(\Gamma)) \). Then, multiply the equation \((\partial_t^2 + \Delta)v_\eta = 0\) by \( w_\eta \) and integrating by parts over \((0, T) \times \Omega\), for any \( \eta \in \mathbb{R}^d \) we have

\[
\int_0^T \int_\Omega (\partial_t^2 - \Delta)v_\eta w_\eta = i \int_\Omega \nabla \cdot (\eta c_1(x)e^{i\eta \cdot x})\beta(x)e^{i\eta \cdot x} \, dx - \int_0^T \int_\Gamma g_\eta \frac{\partial v_\eta}{\partial n} = 0.
\]

Therefore

\[
|\eta|^2 \int_{\Omega'} c_1(x)e^{2i\eta \cdot x} \, dx = \int_0^T \int_\Gamma g_\eta \frac{\partial v_\eta}{\partial n}.
\]

since \( c_1 \equiv 0 \) on \( \Omega \setminus \overline{\Omega'} \). \( \square \)

In term of the function \( v_\eta \) as solution of (11), we introduce

\[
\bar{u}_\alpha(x, t) = u(x, t) + \alpha d \int_0^t e^{-i|\eta|s}v_\eta(x, t-s) \, ds, x \in \Omega, t \in (0, T).
\]

Moreover, for \( z(t) \in C^0_\text{per}([0, T]) \) and for any \( v \in L^1(0, T; L^2(\Omega)) \), we define

\[
\hat{v}(x) = \int_0^T v(x, t)z(t) \, dt \in L^2(\Omega).
\]

The following lemma is useful to proof our main result.

**Lemma 3.1** Consider an arbitrary function \( c(x) \) satisfying condition (3) and assume that conditions (4) and (5) hold. Let \( u, u_\alpha \) be solutions of (8) and (7) respectively. Then, using (16) the following estimates hold:

\[
||u_\alpha - u||_{L^\infty(0, T; L^2(\Omega))} \leq C\alpha,
\]

where \( C \) a positive constant. And,

\[
||\bar{u}_\alpha - u_\alpha||_{L^\infty(0, T; L^2(\Omega))} \leq C'\alpha^{d+1},
\]

where \( C' \) is a positive constant.
Proof. Let \( y_\alpha \) be defined by

\[
\begin{cases}
  y_\alpha \in H^1_0(\Omega), \\
  \Delta y_\alpha = c_\alpha \partial_t (u_\alpha - u) \quad \text{in } \Omega.
\end{cases}
\]

We have

\[
\int_\Omega c_\alpha \partial^2_t (u_\alpha - u) y_\alpha + \int_\Omega \nabla (u_\alpha - u) \cdot \nabla y_\alpha = \alpha \int_\Omega c_1 \nabla u \cdot \nabla y_\alpha.
\]

Since

\[
\int_\Omega \nabla (u_\alpha - u) \cdot \nabla y_\alpha = - \int_\Omega c_\alpha \partial_t (u_\alpha - u)(u_\alpha - u) = -\frac{1}{2} \partial_t \int_\Omega c_\alpha (u_\alpha - u)^2,
\]

and

\[
\int_\Omega c_\alpha \partial^2_t (u_\alpha - u) y_\alpha = \frac{1}{2} \partial_t \int_\Omega |\nabla y_\alpha|^2,
\]

we obtain

\[
\partial_t \int_\Omega |\nabla y_\alpha|^2 + \partial_t \int_\Omega c_\alpha (u_\alpha - u)^2 = -2 \alpha \int_\Omega c_1 \nabla u \cdot \nabla y_\alpha \leq C\alpha |\nabla y_\alpha|_{L^\infty(0,T;L^2(\Omega))}.
\]

From the Gronwall Lemma it follows that

\[
||u_\alpha - u||_{L^\infty(0,T;L^2(\Omega))} \leq C\alpha. \quad \text{(19)}
\]

As a consequence, by using (17) one can see that the function \( \tilde{u}_\alpha - \hat{u} \) solves the following boundary value problem

\[
\begin{cases}
  \Delta (\tilde{u}_\alpha - \hat{u}) = O(\alpha) \quad \text{in } \Omega, \\
  (\tilde{u}_\alpha - \hat{u})\partial \Omega = 0.
\end{cases}
\]

Integration by parts immediately gives,

\[
||(\tilde{u}_\alpha - \hat{u})||_{L^2(\Omega)} = O(\alpha). \quad \text{(20)}
\]

Taking into account that \( (u_\alpha - u) \in L^\infty(0,T;L^2(\Omega)) \), we find by using the above estimate that

\[
||(u_\alpha - u)||_{L^2(\Omega)} = O(\alpha) \quad \text{a.e. } t \in (0,T). \quad \text{(21)}
\]

Under relation (16), one can define the function \( \tilde{y}_\alpha \) as solution of

\[
\begin{cases}
  \tilde{y}_\alpha \in H^1_0(\Omega), \\
  \Delta \tilde{y}_\alpha = c_\alpha \partial_t (\tilde{u}_\alpha - u_\alpha) \quad \text{in } \Omega.
\end{cases}
\]

Integrating by parts immediately yields

\[
\int_\Omega c_\alpha \partial^2_t (\tilde{u}_\alpha - u_\alpha) \tilde{y}_\alpha = -\frac{1}{2} \partial_t \int_\Omega |\nabla \tilde{y}_\alpha|^2,
\]

and

\[
\int_\Omega \nabla (\tilde{u}_\alpha - u_\alpha) \nabla \tilde{y}_\alpha = -\frac{1}{2} \partial_t \int_\Omega c_\alpha (\tilde{u}_\alpha - u_\alpha)^2.
\]

To proceed with the proof of estimate (18), we firstly remark that the function \( \tilde{u}_\alpha \) given by (16) is a solution of

\[
\begin{cases}
  (c_0 \partial^2_t - \Delta) \tilde{u}_\alpha = i a^d \nabla \cdot (\eta c_1(x)e^{i\eta x})e^{-i\eta|\cdot|t} \in L^2(\Omega) \quad \text{in } \Omega \times (0,T), \\
  \tilde{u}_\alpha|_{t=0} = \varphi(x) \quad \text{in } \Omega, \\
  \partial_t \tilde{u}_\alpha|_{t=0} = \psi(x) \quad \text{in } \Omega, \\
  \tilde{u}_\alpha|_{\partial\Omega \times (0,T)} = e^{i\eta x - i\eta|\cdot|t}.
\end{cases}
\]
Then, we deduce that \( u_\alpha - \tilde{u}_\alpha \) solves the following initial boundary value problem,

\[
\begin{aligned}
&\left\{
\begin{array}{l}
(c_\alpha \partial_t^2 - \nabla \cdot \Delta)(u_\alpha - \tilde{u}_\alpha) = \alpha^d \nabla \cdot (c_1(x) \text{ grad } (\int_0^t e^{-i|\eta|s} v_\eta(x, t-s) \, ds)) \quad \text{in } \Omega \times (0, T), \\
(u_\alpha - \tilde{u}_\alpha)|_{t=0} = 0 \quad \text{in } \Omega, \\
\partial_t (u_\alpha - \tilde{u}_\alpha)|_{t=0} = 0 \quad \text{in } \Omega, \\
(u_\alpha - \tilde{u}_\alpha)|_{\partial \Omega \times (0, T)} = 0.
\end{array}
\right.
\end{aligned}
\]

Finally, we can use (22) to find by integrating by parts that

\[
\partial_t \int_\Omega |\nabla \tilde{u}_\alpha|^2 + \partial_t \int_\Omega c_\alpha (\tilde{u}_\alpha - u_\alpha)^2 = 2\alpha^d \int_\Omega c_1 (u - u_\alpha) \cdot \tilde{u}_\alpha
\]

which, from the Gronwall Lemma and by using (21), yields

\[
||\tilde{u}_\alpha - u_\alpha||_{L^\infty(0, T; L^2(\Omega))} \leq C'\alpha^{d+1}.
\]

This achieves the proof. \(\square\)

Now, we identify the function \( c(x) \) by using the difference between local Dirichlet to Neumann maps and the function \( \theta_\eta \) as solution to the Volterra equation (11) or equivalently the ODE (12), as a function of \( \eta \). Then, the following main result holds.

**Theorem 3.1** Let \( \eta \in \mathbb{R}^d \), \( d = 2, 3 \). Suppose that the smooth coefficient \( c(x) \) satisfies (3)-(4)-(5). Let \( u_\alpha \) be the unique solution in \( C^2(0, T; H^1(\Omega)) \cap C^1(0, T; L^2(\Omega)) \) to the wave equation (7) with \( \varphi(x) = e^{iy \cdot x}, \psi(x) = -i|\eta| e^{iy \cdot x} \), and \( f(x, t) = e^{iy \cdot x - i|\eta| t} \). Let \( \tilde{f} \) be the function which satisfies (9). Suppose that \( \Gamma \) and \( T \) geometrically control \( \Omega \), then we have

\[
\int_0^T \int_\Gamma (\theta_\eta + \partial_t \theta_\eta \partial_t \cdot)(\Lambda_\alpha - \Lambda_0)(\tilde{f})(x, t) \, d\sigma(x) dt = \alpha^{d-1}|\eta|^2 \int_\Omega^\prime (c_\alpha - c_0)(x)e^{2iy \cdot x} \, dx + O(\alpha^{d+1}) \quad (23)
\]

\[
= \alpha^d |\eta|^2 \int_\Omega c_1(x)e^{2iy \cdot x} \, dx + O(\alpha^{d+1}),
\]

where \( \theta_\eta \) is the unique solution to the ODE (12) with \( g_\eta \) defined as the boundary control in (10). The term \( O(\alpha^{d+1}) \) is independent of the function \( c_1 \). It depends only on the bound \( M \).

**Proof.** Since the extension of \( (\Lambda_\alpha - \Lambda_0)(\tilde{f})(x, t) \) to \( \partial \Omega \times (0, T) \) is \( \left( \frac{\partial u_\alpha}{\partial n} - \frac{\partial u}{\partial n} \right) \), then by conditions \( \partial_t \theta_\eta(T) = 0 \) and \( (\frac{\partial u_\alpha}{\partial n} - \frac{\partial u}{\partial n})|_{t=0} = 0 \), we have \( (\Lambda_\alpha - \Lambda_0)(\tilde{f})(x, t)|_{t=0} = 0 \). Therefore the term

\[
\int_0^T \int_\Gamma \partial_t \theta_\eta \partial_t (\Lambda_\alpha - \Lambda_0)(\tilde{f})(x, t) \, d\sigma(x) dt
\]

may be simplified as follows

\[
\int_0^T \int_\Gamma \partial_t \theta_\eta \partial_t (\Lambda_\alpha - \Lambda_0)(\tilde{f})(x, t) \, d\sigma(x) dt = -\int_0^T \int_\Gamma \partial_t^2 \theta_\eta (\Lambda_\alpha - \Lambda_0)(\tilde{f})(x, t) \, d\sigma(x) dt. \quad (24)
\]
On the other hand, we have

\[
\int_0^T \int_\Gamma \left[ \theta_{\eta}(\Lambda_{\alpha} - \Lambda_0)(\tilde{f}) + \partial_\eta \theta_{\eta} \partial_t (\Lambda_{\alpha} - \Lambda_0)(\tilde{f}) \right](x, t)d\sigma(x)dt = \int_0^T \int_\Gamma \left[ \theta_{\eta}(\Lambda_{\alpha}(\tilde{f}) - \tilde{\Lambda}_{\alpha}(\tilde{u}_{\alpha}|_{\Gamma \times (0,T)})) + \\
\partial_\eta \theta_{\eta} \partial_t (\Lambda_{\alpha}(\tilde{f}) - \tilde{\Lambda}_{\alpha}(\tilde{u}_{\alpha}|_{\Gamma \times (0,T)})) \right](x, t)d\sigma(x)dt + \\
\int_0^T \int_\Gamma \left[ \theta_{\eta}\alpha^d \int_0^t e^{-i|\eta|s} \frac{\partial v_{\eta}}{\partial n}(x, t - s) ds + \alpha^d \partial_\theta \theta_{\eta} \partial t \int_0^t e^{-i|\eta|s} \frac{\partial v_{\eta}}{\partial n}(x, t - s) ds \right] d\sigma(x)dt;
\]

where \( \tilde{\Lambda}_{\alpha}(\tilde{u}_{\alpha}|_{\Gamma \times (0,T)}) = \Lambda_0(\tilde{f}) + \alpha^d \int_0^t e^{-i|\eta|s} \Lambda_0(\phi)(x, t - s) ds \).

Given that, \( \theta_{\eta} \) satisfies the Volterra equation (12) and

\[
\partial_t (\int_0^t e^{-i|\eta|s} \frac{\partial v_{\eta}}{\partial n}(x, t - s) ds) = \partial_t (-e^{-i|\eta|t} \int_0^t e^{i|\eta|s} \frac{\partial v_{\eta}}{\partial n}(x, s) ds) =
\]

\[
i|\eta|e^{-i|\eta|t} \int_0^t e^{i|\eta|s} \frac{\partial v_{\eta}}{\partial n}(x, s) ds + \frac{\partial v_{\eta}}{\partial n}(x, t),
\]

we obtain by integrating by parts over \((0, T)\) that

\[
\int_0^T \int_\Gamma \left[ \theta_{\eta}(\Lambda_{\alpha} - \Lambda_0)(\tilde{f}) + \partial_\eta \theta_{\eta} \partial_t (\Lambda_{\alpha} - \Lambda_0)(\tilde{f}) \right](x, t)d\sigma(x)dt = \\
\int_0^T \int_\Gamma \left[ \theta_{\eta}(\Lambda_{\alpha}(\tilde{f}) - \tilde{\Lambda}_{\alpha}(\tilde{u}_{\alpha}|_{\Gamma \times (0,T)})) + \partial_\eta \theta_{\eta} \partial_t (\Lambda_{\alpha}(\tilde{f}) - \tilde{\Lambda}_{\alpha}(\tilde{u}_{\alpha}|_{\Gamma \times (0,T)})) \right] d\sigma(x)dt + O(\alpha^{d+1}).
\]

Thus, to prove Theorem 3.1 it suffices then to show that

\[
\int_0^T \int_\Gamma \left[ \theta_{\eta}(\Lambda_{\alpha}(\tilde{f}) - \tilde{\Lambda}_{\alpha}(\tilde{u}_{\alpha}|_{\Gamma \times (0,T)})) + \partial_\eta \theta_{\eta} \partial_t (\Lambda_{\alpha}(\tilde{f}) - \tilde{\Lambda}_{\alpha}(\tilde{u}_{\alpha}|_{\Gamma \times (0,T)})) \right] d\sigma(x)dt = O(\alpha^{d+1}).
\]

From definition (17) we have

\[
\tilde{u}_{\alpha} - \tilde{\alpha}_{\alpha} = \int_0^T (u_{\alpha} - \tilde{u}_{\alpha})z(t) dt,
\]
which gives by system (22) that
\[
\Delta (\tilde{u}_\alpha - \hat{u}_\alpha) = \int_0^T c_\alpha \partial_t^2 (u_\alpha - \tilde{u}_\alpha) z(t) \, dt + \alpha^d \int_0^T \nabla \cdot (c_1(x) \text{ grad } (\tilde{u}_\alpha - \hat{u})) \text{ grad } (\tilde{u}_\alpha - \hat{u}) \, dt.
\]
Thus, by (16) and (22) again, we see that the function \( \tilde{u}_\alpha - \hat{u}_\alpha \) is solution of
\[
\begin{cases}
-\Delta (\tilde{u}_\alpha - \hat{u}_\alpha) = - \int_0^T c_\alpha (u_\alpha - \tilde{u}_\alpha) z''(t) \, dt + \nabla \cdot (c_1(x) \text{ grad } (\tilde{u}_\alpha - \hat{u})) & \text{ in } \Omega, \\
(\tilde{u}_\alpha - \hat{u}_\alpha)|_{\partial \Omega} = 0.
\end{cases}
\]
Taking into account estimate (18) given by Lemma 3.1, then by using standard elliptic regularity (see e.g. [14]) for the boundary value problem (25) we find that
\[
\big| \frac{\partial}{\partial n} (\tilde{u}_\alpha - \hat{u}_\alpha) \big|_{L^2(\Gamma)} = O(\alpha^{d+1}).
\]
The fact that \( \Lambda_\alpha(\tilde{f}) - \tilde{\Lambda}_\alpha(\tilde{u}_\alpha|_{\Gamma \times (0,T)}) := \frac{\partial}{\partial n}(u_\alpha - \tilde{u}_\alpha) \in L^\infty(0,T;L^2(\Gamma)) \), we deduce, as done in the proof of Lemma 3.1, that
\[
||\Lambda_\alpha(\tilde{f}) - \tilde{\Lambda}_\alpha(\tilde{u}_\alpha|_{\Gamma \times (0,T)})||_{L^2(\Gamma)} = O(\alpha^{d+1}),
\]
which implies that
\[
\int_0^T \int_{\Gamma} \left[ \theta_\eta(\Lambda_\alpha(\tilde{f}) - \tilde{\Lambda}_\alpha(\tilde{u}_\alpha|_{\Gamma \times (0,T)})) + \partial_t \theta_\eta \partial_t(\Lambda_\alpha(\tilde{f}) - \tilde{\Lambda}_\alpha(\tilde{u}_\alpha|_{\Gamma \times (0,T)})) \right] d\sigma(x) dt = O(\alpha^{d+1}).
\]
This completes the proof of our Theorem. \( \square \)

We are now in position to describe our identification procedure which is based on Theorem 3.1. Let us neglect the asymptotically small remainder in the asymptotic formula (23). Then, it follows
\[
c_\alpha(x) - c_0(x) \approx \frac{2}{\alpha^{d-1}} \int_{\mathbb{R}^d} e^{-2i\eta \cdot x} \left| \int_0^T \int_{\Gamma} \left( \theta_\eta + \partial_t \theta_\eta \partial_t \cdot \right) (\Lambda_\alpha - \Lambda_0)(\tilde{f})(x,t) d\sigma(y) dt d\eta, x \in \Omega.
\]
The method of reconstruction we propose here consists in sampling values of
\[
\frac{1}{|\eta|^2} \int_0^T \int_{\Gamma} \left( \theta_\eta + \partial_t \theta_\eta \partial_t \cdot \right) (\Lambda_\alpha - \Lambda_0)(\tilde{f})(x,t) d\sigma(x) dt
\]
at some discrete set of points \( \eta \) and then calculating the corresponding inverse Fourier transform.

In the following, it is not hard to prove the more convenient approximation in terms of the values of local Dirichlet-to-Neumann maps \( \Lambda_\alpha \) and \( \Lambda_0 \) at \( \tilde{f} \).

**Corollary 3.1** Let \( \eta \in \mathbb{R}^d \) and let \( \tilde{f} \) be defined by (9). Suppose that \( \Gamma \) and \( T \) geometrically control \( \Omega \), then we have the following approximation
\[
c_\alpha(x) \approx c_0(x) - \frac{2}{\alpha^{d-1}} \int_{\mathbb{R}^d} e^{-2i\eta \cdot x} \left| \int_0^T \int_{\Gamma} \left[ e^{|\eta|t} \partial_h (e^{-|\eta|t} g_\eta(y,t)) (\Lambda_\alpha - \Lambda_0)(\tilde{f})(x,t) \right] d\sigma(y) dt d\eta, x \in \Omega,
\]
where the boundary control \( g_\eta \) is defined by (10).

Proof. The term \( \int_0^T \int_{\Gamma} \partial_t \theta \cdot \partial_t (\Lambda_\alpha - \Lambda_0)(\tilde{f})(x,t) \, d\sigma(x)dt \), given in Theorem 3.1, has to be interpreted as follows:

\[
\int_0^T \int_{\Gamma} \partial_t \theta \cdot \partial_t (\Lambda_\alpha - \Lambda_0)(\tilde{f})(x,t)d\sigma(x)dt = -\int_0^T \int_{\Gamma} \partial_t \theta \cdot (\Lambda_\alpha - \Lambda_0)(\tilde{f})(x,t)d\sigma(x)dt,
\]

because \( \theta |_{t=T} = 0 \) and \( \partial_t (\frac{\partial u}{\partial n} - \frac{\partial u}{\partial n}) |_{t=0} = 0 \). In fact, in view of the ODE (12), the term

\[
\int_0^T \int_{\Gamma} \left[ \theta \cdot \partial_t (\Lambda_\alpha - \Lambda_0) + \partial_t \theta \cdot \partial_t (\Lambda_\alpha - \Lambda_0) \right] \tilde{f}(x,t)d\sigma(x)dt
\]

may be simplified after integration by parts over \((0,T)\) and use of the fact that \( \theta \) is the solution to the ODE (12) to become

\[
-\int_0^T \int_{\Gamma} e^{i|\eta||t} \partial_t (e^{-i|\eta||t} \tilde{g}) \cdot (\Lambda_\alpha - \Lambda_0)(\tilde{f})(x,t)d\sigma(x)dt.
\]

Then, the desired approximation is established. \( \square \)

4 Conclusion

The use of approximate formula (23), including the difference between the local Dirichlet to Neumann maps, represents a promising approach to the dynamical identification and reconstruction of a coefficient which is unknown in a bounded domain (but it is known outside of this domain) for a class of hyperbolic PDE. We believe that this method will yield a suitable approximation to the dynamical identification of small conductivity ball (of the form \( z + \alpha D \)) in a homogeneous medium in \( \mathbb{R}^d \) from the boundary measurements. We will present convenable numerical implementations for this investigation. This issue will be considered in a forthcoming work.

References


[6] M. I. Belishev, Dynamical inverse problem for the equation \( u_{tt} - \Delta u - \nabla n \cdot \nabla u = 0 \) (the BC method), Cubo 10, No. 2, (2008), 15–30.


