Numerical solution of an inverse initial boundary value problem for the full time dependent Maxwell’s equations in the presence of imperfections of small volume

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Abstract

We consider the numerical solution, in a three dimensional bounded domain, of the inverse problem for identifying the location of small volume, electromagnetic imperfections in a medium with homogeneous background. Our new numerical algorithm is based on the coupling of a discontinuous Galerkin finite element solution of the Maxwell’s equations, an exact controllability method, an asymptotic formula and finally a Fourier inversion for localizing the centers of the imperfections. Numerical results, in 3-D, show the robustness and accuracy of the approach for retrieving placed imperfections from boundary measurements.

Key words. Inverse problem, Maxwell’s equations, inhomogeneities, control problem, Fourier Transform

2000 AMS subject classifications. 35R30, 35B40, 35B37, 78M35

1 Introduction

The localization of inhomogeneities in a bounded domain is of great importance since it has several applications: identification of cancer tumors, detection of anti-personnel mines. Recently, several works have been devoted to the numerical analysis of the localization of inhomogeneities (cf [5],[6], [7], [8], [30]). The localization model combines an asymptotic formula with an inversion algorithm. The inversion algorithm is sometimes based on a minimization procedure of least squares type. For the small electric inhomogeneities, the reconstruction is based on a nonlinear minimization procedure. We refer to [27],[29], [10], and [18] for more details about these methods.

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There are a lot of algorithms in the time-domain including time reversal techniques, see [2] for instance and in [3] the authors propose a variety of algorithms in the time-domain including Kirchoff derived using the geometric control method.

More recently, the authors have developed in [9] a numerical method to localize small imperfections for the scalar wave equation using the asymptotic formula established in [5]. But here we consider the full time dependent Maxwell’s equations and we present a new numerical method to localize small electromagnetic inhomogeneities in a bounded domain included in $\mathbb{R}^3$. This procedure generalizes the approach developed in [3] for the two-dimensional (time-independent) inverse conductivity problem and the results in [1] to the full time-dependent Maxwell’s equations. To identify the locations and certain properties of the small inhomogeneities, we use the averaging of the boundary measurements

$$\text{curl } E_{\alpha} \times n|_{\Gamma \times (0,T)};$$

where $E_{\alpha}$ is the electric field in presence of imperfections and $\Gamma$ is a open part of $\partial \Omega$. We present a Fourier type algorithm that derives of a asymptotic formula established in [12] and [14]. We think the best way to numerically validate the localization method and check its robustness, consists of using constructed numerical data. Our attention has been oriented in this direction. We also present an original exact method to solve a controllability problem.

The paper is organised as follows. In the first section, we present the full time dependent Maxwell’s equations and give the framework. Then we recall the theorical results and give the proof of the main result in the third section. In the fourth section, we explain the identification procedure. In the fifth section, we present a discontinuous galerkin method to solve the direct problem since we need to construct numerical data to validate our method. In the next section, we present an original method to solve a control problem which appears in the reconstruction method. In the seventh section, we explain how we use a Fourier inversion for localizing the centers of the imperfections. In the two last sections, we describe the code and give several numerical results about the localization of one and two imperfections in a cube.

2 Presentation of the problem in Electric Field and some notations

Let $\Omega$ be a bounded $C^2$-domain in $\mathbb{R}^3$ which contains a finite number of inhomogeneities, each of the form $z_j + \alpha B_j$, where $B_j \subset \mathbb{R}^3$ is a bounded, smooth domain containing the origin. The total collection of inhomogeneities is $B_\alpha = \bigcup_{j=1}^{m} (z_j + \alpha B_j)$. The points $z_j \in \Omega$, $j = 1, \ldots, m$, are the centers of the inhomogeneities and satisfy the following inequalities:

$$|z_j - z_l| \geq c_0 > 0, \forall j \neq l \quad \text{and} \quad \text{dist}(z_j, \partial \Omega) \geq c_0 > 0, \forall j. \quad (1)$$

Assume that $\alpha > 0$, the common order of magnitude of the diameters of the inhomogeneities, is sufficiently small, that these inhomogeneities are disjoint, and that their
distance to $\mathbb{R}^3 \setminus \Omega$ is larger than $e_0/2$. Let $\mu_0$ and $\varepsilon_0$ denote the permeability and the permittivity of the background medium, and assume that $\mu_0 > 0$ and $\varepsilon_0 > 0$. Let $\mu_j > 0$ and $\varepsilon_j > 0$ denote the permeability and the permittivity of the $j$-th inhomogeneity $z_j + \alpha B_j$ respectively. Let us introduce the piecewise-constant magnetic permeability

$$
\mu_\alpha(x) = \begin{cases} 
\mu_0, & x \in \Omega \setminus \mathcal{B}_\alpha, \\
\mu_j, & x \in z_j + \alpha B_j, \ j = 1 \ldots m.
\end{cases}
$$

The electric permittivity is defined by $\varepsilon_\alpha(x) = \varepsilon_0$, for all $x \in \Omega$. Let $n = n(x)$ denote the outward unit normal vector to $\Omega$ at a point on $\partial \Omega$ and $\Gamma$ an open part of $\partial \Omega$.

![Figure 1: Domain $\Omega$](image)

We will denote by bold letters the functional spaces for the vector fields. Thus $H^s(\Omega)$ denotes the usual Sobolev space on $\Omega$ and $H^s(\Omega)$ denotes $(H^s(\Omega))^3$ and $L^2(\Omega)$ denotes $(L^2(\Omega))^3$. As usual for Maxwell equations, we need spaces of fields with square integrable curls:

$$
H(\text{curl } \Omega) = \{ u \in L^2(\Omega), \text{curl } u \in L^2(\Omega) \},
$$

and with square integrable divergences

$$
H(\text{div } \Omega) = \{ u \in L^2(\Omega), \text{div } u \in L^2(\Omega) \}.
$$

We will also need the following functional spaces:

$$
Y(\Omega) = \{ u \in L^2(\Omega), \text{div } u = 0 \text{ in } \Omega \}, \quad X(\Omega) = H^1(\Omega) \cap Y(\Omega),
$$

and $T L^2(\partial \Omega)$ the space of vector fields on $\partial \Omega$ that lie in $L^2(\partial \Omega)$. Finally, the ”minimal” choice for the electric variational space would be

$$
X_N(\Omega) = \{ v \in H(\text{curl } \Omega) \cap H(\text{div } \Omega); \quad v \times n = 0 \quad \text{on } \partial \Omega \}.
$$

Now, we introduce the following time-dependent Maxwell equations in Electric field

$$
\begin{aligned}
& \left( \varepsilon_\alpha \partial_t^2 + \text{curl } \frac{1}{\mu_\alpha} \text{curl } \right) E_\alpha = 0 \quad \text{in } \Omega \times (0, T), \\
& \text{div } (\varepsilon_\alpha E_\alpha) = 0 \quad \text{in } \Omega \times (0, T), \\
& E_\alpha|_{t=0} = \varphi, \partial_t E_\alpha|_{t=0} = \psi \quad \text{in } \Omega, \\
& E_\alpha \times n|_{\partial \Omega \times (0, T)} = f,
\end{aligned}
$$

(3)
where $E_a \in \mathbb{R}^3$ is the electric field, $f$ the boundary condition for $E_a \times n$, and $\varphi$ and $\psi$ the initial data.

Let $E$ be the solution of the Maxwell's equations in the homogeneous domain:

\[
\begin{cases}
(\varepsilon \partial_t^2 + \text{curl} \frac{1}{\mu_0} \text{curl}) E = 0 & \text{in } \Omega \times (0,T), \\
\text{div} (\varepsilon E) = 0 & \text{in } \Omega \times (0,T), \\
E|_{t=0} = \varphi, \partial_t E|_{t=0} = \psi & \text{in } \Omega, \\
E \times n|_{\partial \Omega \times (0,T)} = f.
\end{cases}
\] (4)

Here $T > 0$ is a final observation time and $\varphi, \psi \in C^\infty(\overline{\Omega})$ and $f \in C^\infty(0,T; C^\infty(\partial \Omega))$ are subject to the compatibility conditions

\[
\partial_t^2 f|_{t=0} = (\Delta^l \varphi) \times n|_{\partial \Omega} \text{ and } \partial_t^{2l+1} f|_{t=0} = (\Delta^l \psi) \times n|_{\partial \Omega}, \quad l = 1, 2, \ldots
\]

it follows that (4) has a unique solution $E \in C^\infty([0,T] \times \overline{\Omega})$. It is also known (see for example [25]) that since $\Omega$ is smooth ($C^2$ - regularity would be sufficient) the non homogeneous Maxwell's equations (3) have a unique weak solution $E_a \in C^0(0,T; X(\Omega)) \cap C^1(0,T; L^2(\Omega))$.

Finally, in [4] the authors have explained that in (4) $f$ and $\psi$ should not be very oscillating.

In the next section, we recall a new asymptotic formula introduced in [12], [13] and [14] for the study of perturbations due to the presence of inhomogeneities.

### 3 Asymptotic formula

We introduce additional notations and definitions. Let us denote $\Phi_j$, $j = 1, \ldots, m$ the unique vector-valued solution to

\[
\begin{cases}
\Delta \Phi_j = 0 & \text{in } B_j, \text{ and } \mathbb{R}^3 \setminus B_j, \\
\Phi_j \text{ is continuous across } \partial B_j, \\
\frac{\mu_j}{\mu_0} \partial_{\nu_j} \Phi_j|_+ - \partial_{\nu_j} \Phi_j|_- = -\nu_j, \\
\lim_{|y| \to +\infty} |\Phi_j(y)| = 0,
\end{cases}
\] (5)

where $\nu_j$ denotes the outward unit normal to $\partial B_j$, and superscripts $-$ and $+$ indicate the limiting values as the point approaches $\partial B_j$ from outside $B_j$, and from inside $B_j$, respectively. The existence and uniqueness of this $\Phi_j$ can be established using single layer potentials with suitably chosen densities, see [10] for the case of conductivity problem. For each inhomogeneity $z_j + \alpha B_j$ we introduce the polarization tensor $M_j$ which is a $3 \times 3$, symmetric, positive definite matrix associated with the $j$-th inhomogeneity, given by

\[
(M_j)_{k,l} = e_k \cdot \left( \int_{\partial B_j} (\nu_j + \frac{\mu_j}{\mu_0} - 1) \frac{\partial \Phi_j}{\partial v_j} (y)) y \cdot e_l \, ds(y) \right).
\] (6)
Here \((e_1, e_2, e_3)\) is an orthonormal basis of \(\mathbb{R}^3\).

First, we give the result about the asymptotic behavior of \(\text{curl } E_\alpha \cdot \nu_j|_{\partial (z_j + \alpha B_j)^+}\).

**Theorem 3.1** Suppose that \((1)\) is satisfied and let \(\Phi_j, j = 1, \ldots, m\) be given as in \((5)\). Then, for the solutions \(E_\alpha, E\) of problems \((3)\) and \((4)\) respectively, and for \(y \in \partial B_j\) we have

\[
\left(\text{curl } E_\alpha(z_j + \alpha y) \cdot \nu_j \right)|_{\partial (z_j + \alpha B_j)^+} = \text{curl } E(z_j, t) \cdot \nu_j + \left(1 - \frac{\mu_j}{\mu_0}\right)\frac{\partial \Phi_j}{\partial \nu_j}|_+ (y) \cdot \text{curl } E(z_j, t) + o(1).
\]

The term \(o(1)\) is uniform in \(y \in \partial B_j\) and \(t \in (0, T)\) and depends on the shape of \(\{B_j\}_{j=1}^m\) and \(\Omega\), the constants \(c_0, T, \mu_0, \{\mu_j\}_{j=1}^m\), the data \(\varphi, \psi, \) and \(f\), but is otherwise independent of the points \(\{z_j\}_{j=1}^m\).

**Proof.** see [12], [13].

We now introduce an auxiliary problem

\[
\begin{cases}
(\partial_t^2 + \text{curl curl })w_\eta = 0 & \text{in } \Omega \times (0, T), \\
\text{div } w_\eta = 0 & \text{in } \Omega \times (0, T), \\
w_\eta|_{t=0} = \beta(x)\eta^2 e^{\eta x}, \partial_t w_\eta|_{t=0} = 0 & \text{in } \Omega, \\
w_\eta \times n|_{\partial \Omega \cap (0, T)} = 0, \\
w_\eta \times n|_{\partial \Omega \cap (0, T)} = g_\eta,
\end{cases}
\]

satisfies \(w_\eta(T) = \partial_t w_\eta(T) = 0\) in \(\Omega\). where \(\beta\) is a cutoff function defined by \(\beta(x) \in C_0^\infty(\Omega)\) such that \(\beta \equiv 1\) in a subdomain \(\Omega'\) of \(\Omega\) that contains the inhomogeneities \(B_\alpha\). And, we denote by \(\eta^\perp\) is a unit vector that is orthogonal to \(\eta\). Then, let \(\theta_\eta \in H^1(0, T; TL^2(\Gamma))\) denote the unique solution of the Volterra equation of second kind

\[
\begin{cases}
\partial_t \theta_\eta(x, t) + \int_0^T e^{-i|\eta|(s-t)}(\theta_\eta(x, s) - i|\eta|\partial_t \theta_\eta(x, s)) \, ds = g_\eta(x, t) & \text{for } x \in \Gamma, t \in (0, T), \\
\theta_\eta(x, 0) = 0 & \text{for } x \in \Gamma.
\end{cases}
\]

The existence and uniqueness of this \(\theta_\eta\) in \(H^1(0, T; TL^2(\Gamma))\) for any \(\eta \in \mathbb{R}^3\) can be established using the resolvent kernel. However, observing from differentiation of \((9)\) with respect to \(t\) that \(\theta_\eta\) is the unique solution of the ODE:

\[
\left\{ \begin{array}{ll}
\partial_t^2 \theta_\eta - \theta_\eta = e^{i|\eta|t}(e^{-i|\eta|t}g_\eta) & \text{for } x \in \Gamma, t \in (0, T), \\
\theta_\eta(x, 0) = 0, \partial_t \theta_\eta(x, T) = 0 & \text{for } x \in \Gamma,
\end{array} \right.
\]

the function \(\theta_\eta\) may be found (in practice) explicitly with variation of parameters and it also immediately follows from this observation that \(\theta_\eta\) belongs to \(H^2(0, T; TL^2(\Gamma))\).

We introduce \(\nu_\eta\) as the unique weak solution (obtained by transposition as done in [23] and in [22] [Theorem 4.2, page 46] for the scalar function) in \(C^0(0, T; X(\Omega)) \cap \)
Proposition 3.1 Suppose that

\[
C^1(0, T; L^2(\Omega))
\]
to the following problem

\[
\begin{align*}
(\partial_t^2 + \text{curl curl}) v_\eta &= 0 \quad \text{in } \Omega \times (0, T), \\
\text{div} \ v_\eta &= 0 \quad \text{in } \Omega \times (0, T), \\
v_\eta|_{t=0} &= 0 \quad \text{in } \Omega, \\
\partial_t v_\eta|_{t=0} &= \sum_{j=1}^{m} i(1 - \frac{\mu_j}{\mu_0}) \eta \times (\nu_j + \frac{\mu_j}{\mu_0} - 1) \frac{\partial \Phi_j}{\partial \nu_j} |_{+}) e^{in z_j} \delta_{\partial(t + \alpha B_j)} \in Y(\Omega) \quad \text{in } \Omega, \\
v_\eta \times n|_{\partial \Omega \times (0, T)} &= 0.
\end{align*}
\]

Then, the following holds.

**Proposition 3.1** Suppose that \( \Gamma \) and \( T \) geometrically control \( \Omega \). For any \( \eta \in \mathbb{R}^3 \) we have

\[
\int_0^T \int_{\Gamma} g_\eta \cdot (\text{curl} \ v_\eta \times n) \ d\sigma(x) \ dt = \alpha^2 \sum_{j=1}^{m} \mu_0(1 - \frac{\mu_j}{\mu_0}) e^{2i\eta z_j} \eta \cdot \int_{\partial B_j} (\nu_j + \frac{\mu_j}{\mu_0} - 1) \frac{\partial \Phi_j}{\partial \nu_j} |_{+}(y) \eta \cdot y \ ds_j(y) + o(\alpha^2).
\]

**Proof.** Multiply the equation \((\partial_t^2 + \text{curl curl}) v_\eta = 0\) by \( w_\eta \) and integrating by parts over \((0, T) \times \Omega\), for any \( \eta \in \mathbb{R}^3 \) we have

\[
\alpha \sum_{j=1}^{m} i(1 - \frac{\mu_j}{\mu_0}) e^{2i\eta z_j} \eta \cdot \int_{\partial B_j} (\nu_j + \frac{\mu_j}{\mu_0} - 1) \frac{\partial \Phi_j}{\partial \nu_j} |_{+}(y) \eta \cdot y \ ds_j(y) = -\mu_0^{-1} \int_0^T \int_{\Gamma} g_\eta \cdot (\text{curl} \ v_\eta \times n) \ d\sigma(x) \ dt.
\]

Now, we take the Taylor expansion of \( \alpha e^{i\eta y} \) in the left side of the last equation, we obtain the convenient asymptotic formula (11).

Then, we give the main result.

**Theorem 3.2** Let \( \eta \in \mathbb{R}^3 \). Let \( E_\eta \) be the unique solution in \( C^0(0, T; X(\Omega)) \cap C^1(0, T; L^2(\Omega)) \) to the Maxwell’s equations (3) with \( \varphi(x) = \eta^1 e^{i\eta x}, \psi(x) = -i \sqrt{\mu_0} |\eta|^2 e^{i\eta z} \), and \( f(x, t) = \eta^1 e^{i\eta x} - i \sqrt{\mu_0} |\eta|^2 \). Suppose that \( \Gamma \) and \( T \) geometrically control \( \Omega \), then we have

\[
\int_0^T \int_{\Gamma} \left[ \theta_\eta \cdot (\text{curl} \ E_\eta \times n) - \text{curl} \ E_\eta \times n \right] \ d\sigma(x) \ dt = \alpha^2 \sum_{j=1}^{m} (\mu_0 - \mu_j) e^{2i\eta z_j} M_j(\eta) \cdot \eta + O(\alpha^2),
\]

where \( \theta_\eta \) is the unique solution to the Volterra equation (10) with \( g_\eta \) defined as the boundary control in (8) and \( M_j \) is the polarization tensor of \( B_j \), defined by

\[
(M_j)_{k,l} = e_k \cdot \left( \int_{\partial B_j} (\nu_j + \frac{\mu_j}{\mu_0} - 1) \frac{\partial \Phi_j}{\partial \nu_j} |_{+}(y) \right) e_l \ ds_j(y).
\]
Here \((e_1, e_2)\) is an orthonormal basis of \(\mathbb{R}^3\). The term \(O(\alpha^2)\) is independent of the points \(\{z_j, j = 1, \cdots, m\}\).

Proof. From \(\partial_t \theta_\eta(T) = 0\) and \((\text{curl } E_\alpha \times n - \text{curl } E \times n)_{|t=0} = 0\) the term \(\int_0^T \int_\Gamma \partial_t \theta_\eta \cdot \partial_t (\text{curl } E_\alpha \times n - \text{curl } E \times n) \, ds (x) \, dt\) has to be interpreted as follows

\[
\int_0^T \int_\Gamma \partial_t \theta_\eta \cdot \partial_t (\text{curl } E_\alpha \times n - \text{curl } E \times n) = - \int_0^T \int_\Gamma \partial_t^2 \theta_\eta \cdot (\text{curl } E_\alpha \times n - \text{curl } E \times n).
\]

Next, introduce

\[
\vec{E}_{\alpha, \eta}(x, t) = E(x, t) + \int_0^t e^{-i\sqrt{\mu_0} |\eta| s} v_\eta(x, t - s) \, ds, x \in \Omega, t \in (0, T).
\]

We have

\[
\int_0^T \int_\Gamma \left[ \theta_\eta \cdot (\text{curl } E_\alpha \times n - \text{curl } E \times n) + \partial_t \theta_\eta \cdot \partial_t (\text{curl } E_\alpha \times n - \text{curl } E \times n) \right] = \\
\int_0^T \int_\Gamma \left[ \theta_\eta \cdot (\text{curl } E_\alpha \times n - \text{curl } \vec{E}_{\alpha, \eta} \times n) + \partial_t \theta_\eta \cdot \partial_t (\text{curl } E_\alpha \times n - \text{curl } \vec{E}_{\alpha, \eta} \times n) \right] + \\
\int_0^T \int_\Gamma \left[ \partial_t \theta_\eta \cdot \int_0^t e^{-i\sqrt{\mu_0} |\eta| s} v_\eta(x, t - s) \times n \, ds + \partial_t \theta_\eta \cdot \partial_t \int_0^t e^{-i\sqrt{\mu_0} |\eta| s} v_\eta(x, t - s) \times n \, ds \right].
\]

Since \(\theta_\eta\) satisfies the Volterra equation (10) and

\[
\partial_t \left( e^{-i\sqrt{\mu_0} |\eta| s} v_\eta(x, t - s) \times n \, ds \right) = \partial_t \left( -e^{-i\sqrt{\mu_0} |\eta| t} \int_0^t e^{i\sqrt{\mu_0} |\eta| s} v_\eta(x, s) \times n \, ds \right)
\]

\[
= i\sqrt{\mu_0} |\eta| e^{-i\sqrt{\mu_0} |\eta| t} \int_0^t e^{i\sqrt{\mu_0} |\eta| s} v_\eta(x, s) \times n \, ds + v_\eta(x, t) \times n,
\]

we obtain by integrating by parts over \((0, T)\) that

\[
\int_0^T \int_\Gamma \left[ \theta_\eta \cdot \int_0^t e^{-i\sqrt{\mu_0} |\eta| s} v_\eta(x, t - s) \times n \, ds + \partial_t \theta_\eta \cdot \partial_t \int_0^t e^{-i\sqrt{\mu_0} |\eta| |t - s|} v_\eta(x, t - s) \times n \, ds \right] = \\
\int_0^T \int_\Gamma \left( v_\eta(x, t) \times n \right) \cdot (\partial_t \theta_\eta + \int_0^t \theta_\eta(s) e^{i\sqrt{\mu_0} |\eta| (t - s)} ds) \\
- i\sqrt{\mu_0} |\eta| (e^{-i\sqrt{\mu_0} |\eta| t} \partial_t \theta_\eta(t)) \cdot \int_0^t e^{i\sqrt{\mu_0} |\eta| s} v_\eta(x, s) \times n \, ds \, dt \\
= \int_0^T \int_\Gamma v_\eta(x, t) \times n \cdot (\partial_t \theta_\eta + \int_0^T (\theta_\eta(s) - i\sqrt{\mu_0} |\eta| \partial_t \theta_\eta(s)) e^{i\sqrt{\mu_0} |\eta| (t - s)} ds) dt \\
= \int_0^T \int_\Gamma g_\eta(x, t) \cdot (\text{curl } v_\eta(x, t) \times n) \, dt
\]
and so, from Proposition 3.1 we obtain

\[ \int_0^T \int_{\Gamma} \left[ \theta_\eta \cdot (\text{curl } E_\alpha \times \mathbf{n} - \text{curl } E \times \mathbf{n}) + \partial_t \theta_\eta \cdot \partial_t (\text{curl } E_\alpha \times \mathbf{n} - \text{curl } E \times \mathbf{n}) \right] = \]

\[ \alpha^2 \sum_{j=1}^m (1 - \frac{\mu_j}{\mu_0}) e^{2\eta z_j} \cdot \int_{\partial B_j} (\nu_j + (\frac{\mu_j}{\mu_0} - 1) \frac{\partial \Phi_j}{\partial \nu_j}|_{\partial \Omega}) \eta \cdot y \, ds_j(y) \]

\[ + \int_0^T \int_{\Gamma} \left[ \theta_\eta \cdot (\text{curl } E_\alpha \times \mathbf{n} - \text{curl } \tilde{E}_{\alpha,\eta} \times \mathbf{n}) + \partial_t \theta_\eta \cdot \partial_t (\text{curl } E_\alpha \times \mathbf{n} - \text{curl } \tilde{E}_{\alpha,\eta} \times \mathbf{n}) \right] + o(\alpha^2). \]

In order to prove Theorem 3.2 it suffices then to show that

\[ \int_0^T \int_{\Gamma} \left[ \theta_\eta \cdot (\text{curl } E_\alpha \times \mathbf{n} - \text{curl } \tilde{E}_{\alpha,\eta} \times \mathbf{n}) + \partial_t \theta_\eta \cdot \partial_t (\text{curl } E_\alpha \times \mathbf{n} - \text{curl } \tilde{E}_{\alpha,\eta} \times \mathbf{n}) \right] = o(\alpha^2). \]

(16)

Since

\[
\left\{ \begin{array}{l}
(\partial_t^2 - \text{curl} \frac{1}{\mu_0} \text{curl})(\int_0^t e^{-i\sqrt{\mu_0}|\eta|s} v_\eta(x, t - s) \, ds)

= \sum_{j=1}^m (1 - \frac{\mu_j}{\mu_0}) \eta \times (\nu_j + (\frac{\mu_j}{\mu_0} - 1) \frac{\partial \Phi_j}{\partial \nu_j}|_{\partial \Omega}) e^{iy z_j} \delta_{\theta(z_j + \alpha B_j)} e^{-i\sqrt{\mu_0}|\eta|t} \quad \text{in } \Omega \times (0, T),

(\int_0^t e^{-i\sqrt{\mu_0}|\eta|s} v_\eta(x, t - s) \, ds)|_{t=0} = 0, \partial_t(\int_0^t e^{-i\sqrt{\mu_0}|\eta|s} v_\eta(x, t - s) \, ds)|_{t=0} = 0 \quad \text{in } \Omega,

(\int_0^t e^{-i\sqrt{\mu_0}|\eta|s} v_\eta(x, t - s) \, ds) \times \mathbf{n}|_{\partial \Omega \times (0, T)} = 0,
\end{array} \right. \]

it follows from Theorem 3.1 that

\[
\left\{ \begin{array}{l}
(\partial_t^2 - \text{curl} \frac{1}{\mu_0} \text{curl})(E_\alpha - \tilde{E}_{\alpha,\eta}) = o(\alpha^2) \quad \text{in } \Omega \times (0, T),

(E_\alpha - \tilde{E}_{\alpha,\eta})|_{t=0} = 0, \partial_t(E_\alpha - \tilde{E}_{\alpha,\eta})|_{t=0} = 0 \quad \text{in } \Omega,

(E_\alpha - \tilde{E}_{\alpha,\eta}) \times \mathbf{n}|_{\partial \Omega \times (0, T)} = 0.
\end{array} \right. \]

We can easily obtain

\[ ||E_\alpha - \tilde{E}_{\alpha,\eta}||_{L^2(\Omega)} = o(\alpha^2), t \in (0, T), x \in \Omega, \]

where \(o(\alpha^2)\) is independent of the points \(\{z_j\}_{j=1}^m\). To prove (16) it suffices then from (14) to show that the following estimate holds

\[ ||\text{curl } E_\alpha \times \mathbf{n} - \text{curl } \tilde{E}_{\alpha,\eta} \times \mathbf{n}||_{L^2(0, T; TL^2(\Omega))} = o(\alpha^2). \]

Let \(\theta\) be given in \(C_0^\infty[0, T]\) and define

\[ \tilde{E}_{\alpha,\theta}(x) = \int_0^T \tilde{E}_{\alpha,\theta}(x, t) \theta(t) \, dt \]
and
\[ \hat{E}_\alpha(x) = \int_0^T E_\alpha(x, t) \theta(t) \, dt. \]

From definition (15) we can write
\[
\begin{aligned}
(\hat{E}_\alpha - \tilde{E}_\alpha) &\in \mathbf{H}^1(\Omega), \\
\text{curl} \ \text{curl} (\hat{E}_\alpha - \tilde{E}_\alpha) &= 0(\alpha) \in Y(\Omega) \quad \text{in } \Omega, \\
\text{div} (\hat{E}_\alpha - \tilde{E}_\alpha) &= 0 \quad \text{in } \Omega, \\
(\hat{E}_\alpha - \tilde{E}_\alpha) \times \mathbf{n}|_{\partial\Omega} &= 0.
\end{aligned}
\]

(17)

In the spirit of the standard elliptic regularity [17] and we can also see [12] where there is a similar demonstration, we deduce for the boundary value problem (17) that
\[
|| \text{curl} (\hat{E}_\alpha - \tilde{E}_\alpha) \times \mathbf{n}||_{\mathbf{L}^2(\Gamma)} = O(\alpha^2),
\]
for all \( \theta \in C_0^\infty([0, T]) \); whence
\[
|| \text{curl} (E_\alpha - \hat{E}_\alpha) \times \mathbf{n}||_{\mathbf{L}^2(\Gamma)} = o(\alpha^2) \text{ a.e. in } t \in (0, T),
\]
and so, the desired estimate (12) holds. The proof of Theorem 3.2 is then over. \( \Box \)

In the next section, we briefly explain the reconstruction method. It is based on Theorem 3.2.

4 Identification procedure

We define the function \( N_\alpha(\eta) \) by
\[
N_\alpha(\eta) = \int_0^T \int_{\Gamma} \left[ \theta_\eta \cdot \left( \text{curl} (E_\alpha - E) \times \mathbf{n} \right) + \partial_t \theta_\eta \cdot \partial_t \left( \text{curl} (E_\alpha - E) \times \mathbf{n} \right) \right] d\sigma(x) dt.
\]

Then if we neglect the remainder in the asymptotic formula (12), according to theorem 3.2, the Fourier Transform of \( N_\alpha(\eta) \) is
\[
\tilde{N}_\alpha(x) \approx \alpha^2 \sum_{j=1}^m L_j \delta_{-2z_j},
\]
where \( L_j \) is a second order constant coefficient, differential operator whose coefficients depend on the polarization tensor \( M_j \) defined in (13) and \( \tilde{N}_\alpha(x) \) represents the inverse Fourier Transform of \( N_\alpha(\eta) \) and \( \delta_{-2z_j} \) is the the Dirac function at point \(-2z_j\).

The method of reconstruction consists in sampling values of \( \tilde{N}_\alpha(x) \) at some discrete set of points and then calculating the corresponding discrete inverse Fourier Transform. After a rescaling the support of this discrete inverse Fourier Transform yields the location of the small inhomogeneities \( B_\alpha \). Once the locations are known we may calculate the polarization tensors \( (M_j)_{j=1}^m \) by solving an appropriate linear system arising from (12).

We need to know \( \text{curl} (E_\alpha - E) \times \mathbf{n} \). Therefore, we present a discontinuous Galerkin method to solve the direct problem (3) which has been studied in [15] and [16].
5 Discontinuous Formulation for the direct problem

5.1 Discontinuous formulation for the wave equation

First, let us name \( u \) the perturbation of the electric field \( E_\alpha - E \), then the Maxwell’s system on perturbation is given by

\[
\begin{aligned}
\partial_t^2 u + \text{curl} \frac{1}{\mu_0 \varepsilon_0} \text{curl} u &= (1 - \frac{\mu_0}{\mu_\alpha}) \partial_t^2 E \quad \text{in } \Omega \times (0, T), \\
\text{div} (\varepsilon_0 u) &= 0 \quad \text{in } \Omega \times (0, T), \\
u|_{t=0} = 0, \partial_t u|_{t=0} &= 0 \quad \text{in } \Omega, \\
u \times n|_{\partial\Omega \times (0, T)} &= 0.
\end{aligned}
\]  
(18)

with

\[ E(x, t) = \eta^i e^{i\eta \cdot x - ic|\eta| t} \]

and

\[ \mu_\alpha(x) = \begin{cases} 
\mu_0, & \text{in } \Omega \setminus \mathring{B}_h, \\
\mu_j, & \text{in } z_j + \alpha B_j, \quad j = 1 \ldots m.
\end{cases} \]

Let \( \Pi_h \) be a shape regular triangulation of \( \Omega \) by tetrahedra. We suppose that \( \mu_\alpha \) is constant by element of triangulation \( \Pi_h \) and to simplify our formulation we consider the permittivity constant \( \varepsilon_0 = 1 \).

If \( K \in \Pi_h \), we denote by \( h_K \) the diameter of \( K \). We define the average, tangential and normal jumps of \( w \) at \( x \in e \) an internal face by

\[
\{\omega\} = \frac{\omega_1 + \omega_2}{2}, \quad [\omega]_T = n_1 \times \omega_1 + n_2 \times \omega_2, \quad [\omega]_N = n_1 \cdot \omega_1 + n_2 \cdot \omega_2 \]

and if \( e \subset \partial \Omega \), we set for \( \{\omega\} = \omega, [\omega]_T = n \times \omega \) and \( [\omega]_N = n \cdot \omega \).

We propose the variational problem:

\[
(u_{tt}, v) + B(u, v) = (f(u), v) \quad \forall v \in H^1(\nabla \times, \Pi_h). 
\]  
(19)

with

\[
B(u, v) = (\frac{1}{\mu_\alpha} \nabla \times u, \nabla \times v) + < n \times \frac{1}{\mu_\alpha} (\nabla \times u), v > \\
- \frac{1}{\mu_\alpha} \sum_{e \subset F^t_h} < \nabla \times u, \nabla \times v >_e + < n \times \frac{1}{\mu_\alpha} (\nabla \times u), u > \\
- \frac{1}{\mu_\alpha} \sum_{e \subset F^t_h} < \nabla \times u, \nabla \times v >_e + (\nabla \cdot u, \nabla \cdot v) \\
+ \sum_{e \subset F^t_h} < \sigma |u|_N, [v]_N >_e + \sum_{e \subset F^l_h} < \sigma |u|_T, [v]_T >_e, \quad u, v \in H^1(\Pi_h)^3
\]

where \( \sigma \) is a stabilization parameter and will be chosen depending on the local meshsize, polynomial degree and \( \mu_\alpha(K), \quad K \in \Pi_h \) and

\[ H^1(\nabla \times, \Pi_h) := \{ v : v|_K \in H^s(K)^3 \text{ and } \nabla \times (v|_K) \in H^s(K)^3, \forall K \in \Pi_h \}. \]

We have the following a priori error estimate [15] and [16].
Proposition 5.1. Let $\mu_K = \min(p_K + 1, t_K)$ and $u$ be the exact solution of (1)-(2). Suppose that $u_{iK} \in C^2(I, H^1(K))$, $\forall K \in \Pi_h$ with $t_K \geq 2$. Let $U$ the discrete solution of (5). Then, the error $\zeta = U - u$ satisfies

$$\|\zeta(t)\|^2 + \|\zeta'(t)\|^2 + J^p(\zeta(t), \zeta(t)) \leq C \sum_{K \in \Pi_h} \frac{h^{2\mu_K - 2}}{p^{2H-2}} \left( \|u_t\|_{L^2(H^1(K))^3} + \|u\|_{L^\infty(H^2(K))^3} + \|u_t\|_{L^2(H^2(K))^3} + \|u\|_{L^\infty(H^2(K))^3} \right).$$

5.2 Approximation of the DG formulation

We approximate the problem (19) with $\Sigma_h = S^1(\Pi_h)^3$ such that

$$S^1(\Pi_h) = \{ u \in L^2(\Omega) : u_{iK} \in S^1(K), \forall K \in \Pi_h \}$$

where $S^1(K)$ is the space of real polynomials of degree at most 1 in $K$. The time discretization of the problem (19) in space by the DG method leads to the linear second order system of ordinary differential equations

$$M\dot{u}^h(t) + A u^h(t) = f^h(t), \quad t \in I$$

with initial conditions

$$M u^h(0) = u_0^h, \quad M \dot{u}^h(0) = u_1^h.$$  \hfill (21)

Here, $M$ denotes the mass matrix and $A$ the stiffness matrix. To discretize (20) in time, we employ the Newmark time stepping scheme; see, e.g. [28]. We let $k$ denote the time step and set $t_n = n \cdot k$. Then the Newmark method consists in finding approximation $\{u^h_n\}_n$ to $u^h(t_n)$ such that

$$(M + k^2\beta A)u^h_1 = \left[ M - k^2 \left( \frac{1}{2} - \beta \right) A \right] u^h_0 + kM u^h_1 + k^2 \left[ \beta f^1_n + \left( \frac{1}{2} - \beta \right) f^0_n \right]$$

and

$$(M + k^2\beta A)u^h_n = \left[ M - k^2 \left( \frac{1}{2} - 2\beta + \gamma \right) A \right] u^h_{n-1}$$

$$- \left[ M + k^2 \left( \frac{1}{2} + \beta - \gamma \right) A \right] u^h_n$$

$$+ k^2 \left[ \beta f^h_{n+1} + \left( \frac{1}{2} - 2\beta + \gamma \right) f^h_n + \left( \frac{1}{2} - 2\beta + \gamma \right) f^h_{n-1} \right],$$

for $n = 1, 2, \ldots N - 1$. Here $f_n := f(t_n)$ while $\beta \geq 0$ and $\gamma \geq \frac{1}{2}$ are free parameters that still can be chosen. We recall that for $\gamma = \frac{1}{2}$ the Newmark scheme is second order accurate in time, whereas it is only first order accurate for $\gamma > \frac{1}{2}$. For $\beta = 0$, the Newmark scheme (22)-(23) requires at each time step the solution of a linear system with the matrix $M$.

In our test, we will employ the both implicit second order Newmark scheme, setting $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{2}$ in (22)-(23), and the explicit one with $\gamma = \frac{1}{2}$ and $\beta = 0$ in (22)-(23).

Now, we calculate $\theta_\eta$ to evaluate $\kappa_\eta(\eta)$. First, we solve a control problem to obtain the function $g_\eta$ then we solve the EDO (10) with Runge-Kutta to obtain the function $\theta_\eta$. 

11
6 Control problem

6.1 Control problem formulation

We check a function \( g \) such as the unique weak solution \( w_\eta \) of

\[
\begin{align*}
\frac{\partial^2}{\partial t^2} + \text{curl} \text{ curl } w_\eta &= 0 \quad \text{in } \Omega \times (0, T), \\
\text{div } w_\eta &= 0 \quad \text{in } \Omega \times (0, T), \\
w_\eta|_{t=0} &= \beta(x)\eta \cdot e^{i\eta \cdot x}, \partial_tw_\eta|_{t=0} = 0 \quad \text{in } \Omega, \\
w_\eta \times n|_{\partial\Omega T \times (0, T)} &= 0, \\
w_\eta \times n|_{\Gamma \times (0, T)} &= g_\eta,
\end{align*}
\]

(24)

satisfies \( w_\eta(T) = \partial_t w_\eta(T) = 0 \) in \( \Omega \).

It is natural that to obtain the controllability of the solution of the problem (24) we have to verify two following conditions:

- \( T \) must be rather big (the wave propagation speed is finite).
- \( \Gamma \) must verify the Geometric Control Condition. (see [20] and [21] for more details over the Geometric Control)

The second condition could be also given by the following theorem:

**Theorem 6.1** (Bardos-Lebeau-Rauch, 1992) In the case where all possible geodesic lines of the length \( T \) reach the controlled boundary \( \Gamma \) at the non-diffractive point then for all \((u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)\) there exists a control function \( g_\eta \in L^2(\Gamma \times (0, T)) \) such as the unique solution of the problem (24) verifies the controllability criteria

\[
u(x, T) = \partial_t u(x, T) = 0 \quad \text{in } \Omega.
\]

If the conditions are verified then there exist unique solution \( g_\eta \in L^2(\Gamma \times (0, T)) \) of the control problem given by (24).

We present an original method to obtain the unknown function \( g_\eta \) analytically.

6.2 Plane waves decomposition method

Let us consider the the cutoff function \( \beta(x) \in C_0^\infty(\Omega) \) such that \( \beta \equiv 1 \) in a subdomain \( \Omega' \) of \( \Omega \) that contains the inhomogeneities \( \Sigma_\eta \). It is very easy to show that if we take \( w_\eta(x, t) = u_\eta(\eta \cdot x + c_0|\eta|t) + v_\eta(\eta \cdot x - c_0|\eta|t) \), where \( u_\eta \) and \( v_\eta \) are the plane waves with positive and negative time directions such as

\[
u_\eta(x, t) = \frac{1}{2} \beta(\eta \cdot x + c_0|\eta|t) \cdot \eta^\perp e^{i(\eta \cdot x + c_0|\eta|t)}
\]

and

\[
v_\eta(x, t) = \frac{1}{2} \beta(\eta \cdot x - c_0|\eta|t) \cdot \eta^\perp e^{i(\eta \cdot x - c_0|\eta|t)},
\]

then the constructed function \( w_\eta(x, t) \) will be the solution of (24).
Moreover, we can show immediately that such solution will be controllable if the final time $T$ is sufficiently big, cf [19] for example. To explain our approach let’s see the illustration given for the one dimensional case (Fig. 2 - Fig. 3). As we can observe, $w_\eta(x,T) = 0$ at the time $T$, and we obtain for the time derivatives

$$\partial_t w_\eta(x,T) = \partial_t u_\eta(x,t) + \partial_t v_\eta(x,t)$$

$$\partial_t u_\eta(x,t) = \frac{1}{2} c_0 |\eta|(\beta'(\eta \cdot x + c_0 |\eta|t) \cdot \eta^\perp e^{i(\eta \cdot x + c_0 |\eta|t)}) + \beta(\eta \cdot x + c_0 |\eta|t) \cdot \eta^\perp e^{i(\eta \cdot x + c_0 |\eta|t)})$$

and

$$\partial_t v_\eta(x,t) = -\frac{1}{2} c_0 |\eta|(\beta'(\eta \cdot x - c_0 |\eta|t) \cdot \eta^\perp e^{i(\eta \cdot x - c_0 |\eta|t)}) + \beta(\eta \cdot x - c_0 |\eta|t) \cdot \eta^\perp e^{i(\eta \cdot x - c_0 |\eta|t)})$$

and the condition $\partial_t w_\eta(x,T) = 0$ is verified.
6.2.1 Numerical results

We briefly give the results of the numerical validation test of the presented method. The domain $\Omega$ is the cube with dimensions $[0,1]^3$ contains 1034 tetrahedra, $T = 1.7s$ and the time step number was taken equal to 30. We show at Fig. 4 and in Table 1 $||w(x,t)||_{L^2(\Omega)}$ for the last five time steps $t = [26\Delta t \ldots 30\Delta t]$ and at Fig. 5 $||\partial_t w(x,t)||_{L^2(\Omega)}$ for the same time steps.

![Figure 4: $||w(x,t)||_{L^2(\Omega)}$ for last five time steps](image)

![Figure 5: $||\partial_t w(x,t)||_{L^2(\Omega)}$ for last five time steps](image)

7 IFFT algorithm

In this section we observe the numerical implementation of the Fourier transform and recall its proprieties. This method has also been used in [30].
To obtain the localization of the inhomogeneities centers we need to apply a Fourier inversion transform to the function \( \mathcal{K}_\alpha(\eta) \) given by

\[
\mathcal{K}_\alpha(\eta) = \int_0^T \int_{\Gamma} \left[ \theta_\eta \cdot (\text{curl} (E_\alpha - E) \times \mathbf{n}) + \partial_t \theta_\eta \cdot \partial_t (\text{curl} (E_\alpha - E) \times \mathbf{n}) \right] \, d\sigma(x) \, dt.
\]

We have to recall also that the function \( e^{2i\eta \cdot z_j} \) is exactly the Fourier Transform (up to a multiplicative constant) of the Dirac function \( \delta_{-2z_j} \) (a point mass located at \(-2z_j\)), where the set of the points \( z_j, j = 1, \ldots, m \) represents the centers of the inclusions to be detected. As well, if we consider that we have already constructed numerically the \( \mathcal{K}_\alpha(\eta) \), after applying the IFFT (Inverse Fast Fourier Transforms) algorithm over \( \mathcal{K}_\alpha(\eta) \), we obtain the linear combination of the Dirac functions \( \delta_{-2z_j} \). And then, after rescaling, we obtain the total collection of the points \( z_j, j = 1, \ldots, m \). However, it has to be noted here that the number of different values of the variable \( \eta \in \mathbb{R}^3 \), which is considered as the set of the frequencies for the Inverse Fourier Transform, is very important for the total computation time for the direct problem, and also gives the final detection precision.

From the Shannon’s theorem two principal facts follow:

- If the domain \( \Omega \) which contains the inclusions is a cube with the dimension \( M \) then the function \( \mathcal{K}_\alpha(\eta) \) has to be sampled with the step size \( \Delta \eta = \frac{1}{M} \).
- If we sample in the frequency domain \( |\eta| < \eta_{\text{max}} \) then the reconstruction resolution will not be less then \( \delta = \frac{1}{2\eta_{\text{max}}} \).

Moreover, we have to be in possession of the \( N_\delta = (\frac{M}{\delta})^3 \), for the three dimensional case, sampled values of \( \mathcal{K}_\alpha(\eta) \) to reach the resolution \( \delta \) of the reconstruction of the inclusion’s centers displaced in the cube of the dimensions \( M \). It follow immediately that we have to proceed to the sampling at least of \( (2M\eta_{\text{max}})^3 \) sampling points.

We follow the approach explained in [30]. Then, we need to calculate correctly the frequency step size \( \Delta \eta \). If we consider that the domain \( \Omega \) contains the inclusions is in the form of the square (or cube in three dimensional case) with the dimension \( M \), then we can obtain the estimate

\[
\frac{M\Delta \eta}{2\delta} \approx \frac{1}{3}.
\]
Such method of sampling parameters calculation will be used for our numerical simulations. The results will be given in the next section.

8 Computational program structure

We give the general explanation of the structure of our modelling program:

- **Direct problem solution:**
  - Fortran 90 computation code development for 3D model.
  - The $hp$–discontinuous Galerkin method.

- **The boundary control problem:**
  - Base vectors decomposition. (Decomposition by 1-dimensional waves for the positive and negative time orientation.) As it was described earlier.

- **The Volterra equation resolution** using classical Runge-Kutta method.

- $\mathcal{N}_\alpha(\eta)$ computation.

- Reconstruction of the dislocations of the inhomogeneities using IFFT (Matlab).

- Post-processing module for the inhomogeneity’s center detection and the representing of the results (Matlab).

As it was mentioned before, we need to solve the direct problem and realize $\mathcal{N}_\alpha(\eta)$ computation for each value of the variable $\eta$. The obtained matrix has to be exported to the Matlab post-processing module to apply the IFFT and to visualize the obtained data.

9 Numerical results

In this section we give the numerical results. We consider a regular triangulations of $\Omega$ in tetrahedra. In our study, $\Omega$ is the cube $[0,1]^3$. The triangulation contains 1034 tetrahedra. The explicit Newmark scheme (22)–(23) has been used to solve the direct problem. Then, we apply the discrete Fourier transform to the function $\mathcal{N}_\alpha(\eta)$. We also obtain the center of inclusions computing the maximal value of the matrix.

9.1 Single inclusion case tests

We study different localization of one inclusion (Fig. 6) and for different values of $\eta$. 
We present here two tests:

1. in the first test, we put the imperfection at the point \((0.4; 0.4; 0.4)\), see table 2 for the data,

2. in the second test, we displace the imperfection in \((0.3; 0.3; 0.3)\), see table 3.

<table>
<thead>
<tr>
<th>Mesh</th>
<th>Cube ([0..1]^3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of elements</td>
<td>1034</td>
</tr>
<tr>
<td>Newmark parameters</td>
<td>(\gamma = \frac{1}{2}, \beta = 0)</td>
</tr>
<tr>
<td>Time steps number</td>
<td>800</td>
</tr>
<tr>
<td>Total time</td>
<td>(T = 1.7s)</td>
</tr>
<tr>
<td>Number of (\eta)</td>
<td>7</td>
</tr>
<tr>
<td>Inclusion center</td>
<td>((0.4; 0.4; 0.4))</td>
</tr>
<tr>
<td>Inclusion diameter</td>
<td>0.08</td>
</tr>
<tr>
<td>Center detected ((max(\mathcal{K}_\eta(\eta))))</td>
<td>((0.428; 0.428; 0.428))</td>
</tr>
<tr>
<td>Figure</td>
<td>Fig. 7, Fig. 8</td>
</tr>
</tbody>
</table>

Table 2: Single inclusion Test 1
Mesh | Cube $[0.1]^3$
---|---
Number of elements | 1034
Newmark parameters | $\frac{1}{4}$, $\beta = 0$
Time steps number | 800
Total time | $T = 1.7s$
Number of $\eta$ | 11
Inclusion center | $(0.3; 0.3; 0.3)$
Inclusion diameter | 0.12
Center detected ($\max(N_u(\eta))$) | $(0.363; 0.363; 0.363)$
Reconstruction precision | 0.09
Figure | Fig. 9, Fig. 10

Table 3: Single inclusion Test 2

Figure 7: Test 1 ($X, Y$), ($X, Z$)

Figure 8: Test 1 ($Y, Z$), $Mesh$
The tables 2 and 3 show the accuracy of the detection and the CPU time is very short in the two tests. We can see for the test 1, that the maximal value of the elements in the three dimensional matrix, obtained by the Fourier inversion transform is located at the point $(0.428; 0.428; 0.428)$, while the center point of the inclusion is $(0.4; 0.4; 0.4)$. Similarly we see in test 2, the localization of the imperfection is $(0.363; 0.363; 0.363)$ instead of $(0.3; 0.3; 0.3)$. This shows that the method is accurate. We can see from Fig. 7, Fig. 8, Fig 9 and Fig 10 the center of the peak corresponds to the center of the inclusion introduced in $\Omega$.

Then, we present numerical results for different values of $\eta$: $\eta = 7, 9$ and 11. The table 4 give the data. We can see that when the number of $\eta$ increases the method is really more accurate.
Table 4: Dependence of the number of $\eta$

<table>
<thead>
<tr>
<th>Mesh</th>
<th>Number of $\eta = 7$</th>
<th>Number of $\eta = 9$</th>
<th>Number of $\eta = 11$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of elements</td>
<td>Cube $[0,1]^3$</td>
<td>Cube $[0,1]^3$</td>
<td>Cube $[0,1]^3$</td>
</tr>
<tr>
<td>Newmark parameters</td>
<td>$\gamma = \frac{1}{2}$, $\beta = \frac{1}{2}$</td>
<td>$\gamma = \frac{1}{2}$, $\beta = \frac{1}{2}$</td>
<td>$\gamma = \frac{1}{2}$, $\beta = \frac{1}{2}$</td>
</tr>
<tr>
<td>Time steps number</td>
<td>30</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>Total time</td>
<td>$T = 1.0s$</td>
<td>$T = 1.0s$</td>
<td>$T = 1.0s$</td>
</tr>
<tr>
<td>Inclusion center</td>
<td>(0.4; 0.4; 0.4)</td>
<td>(0.4; 0.4; 0.4)</td>
<td>(0.4; 0.4; 0.4)</td>
</tr>
<tr>
<td>Inclusion diameter</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>Center detected ($max(N,\eta)$)</td>
<td>(0.428; 0.428; 0.428)</td>
<td>(0.4; 0.4; 0.4)</td>
<td>(0.464; 0.464; 0.464)</td>
</tr>
<tr>
<td>Figure</td>
<td>Fig. 11</td>
<td>Fig. 12</td>
<td>Fig. 13</td>
</tr>
</tbody>
</table>

Figure 11: $Nb_\eta = 7, (X, Y), (X, Z)$

Figure 12: $Nb_\eta = 9, (X, Y), (X, Z)$
Double inclusion case test

In this part we show the results of the modelling for the double inclusion case. We put two inclusions with the same constant $\mu_\alpha$ value in $\Omega$, always represented by the cube $[0..1]^3$ with respect of the separation conditions (1). The data are in the table 5.

<table>
<thead>
<tr>
<th>Mesh</th>
<th>Cube $[0..1]^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of elements</td>
<td>1034</td>
</tr>
<tr>
<td>Newmark parameters</td>
<td>$\gamma = \frac{1}{2}$, $\beta = 0$</td>
</tr>
<tr>
<td>Time steps number</td>
<td>800</td>
</tr>
<tr>
<td>Total time</td>
<td>$T = 1.7s$</td>
</tr>
<tr>
<td>Number of $\eta$</td>
<td>7</td>
</tr>
<tr>
<td>First inclusion center</td>
<td>(0.6; 0.3; 0.6)</td>
</tr>
<tr>
<td>Second inclusion center</td>
<td>(0.6; 0.8; 0.6)</td>
</tr>
<tr>
<td>First inclusion diameter</td>
<td>0.1</td>
</tr>
<tr>
<td>Second inclusion diameter</td>
<td>0.1</td>
</tr>
<tr>
<td>First center detected ($max(\mathcal{N}_\alpha(\eta))$)</td>
<td>(0.625; 0.375; 0.625)</td>
</tr>
<tr>
<td>Second center detected ($max(\mathcal{N}_\alpha(\eta))$)</td>
<td>(0.625; 0.750; 0.625)</td>
</tr>
<tr>
<td>Reconstruction precision</td>
<td>0.14</td>
</tr>
</tbody>
</table>

Table 5: Double inclusion case test
We can see that the method permits to localize the imperfections with accuracy. The CPU time is always very small.

10 Conclusion

Very promising approximate formula for the dynamical identification of small inhomogeneities embedded in a homogeneous medium have been recalled for the full time-depended Maxwell’s equation. The theoretical base for the precision reconstruction method of small inhomogeneities location has been confirmed by numerical simulations. The present reconstruction method could be easily adopted for the different geometry of the inhomogeneities as it was shown by the different numerical simulations.

The numerical computational code has been developed. It could be divided in two principal parts, first of all, the numerical resolution of full time-depended Maxwell’s equation using $hp$-discontinuous Galerkin Method in space and Newmark scheme in
time for the reconstruction problem’s statement data simulation, and second part, the inversion using Fourier transforms algorithm of the constructed boundary integral, which gives the inclusion’s centers reconstruction. All of them, the theoretical result and computational program have been validated by the modelling results.

References


