A $hp-$ discontinuous Galerkin method for the time-dependent Maxwell's equation : a priori error estimate

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Abstract. A discontinuous Galerkin method for the numerical approximation for the time-dependant Maxwell's equations in "stable medium" with supraconductive boundary, is introduced and analysed. its $hp-$analysis is carried out and error estimates that are optimal in the meshsize h and slightly suboptimal in the approximation degree p are obtained.

Keywords: Discontinuous Galerkin method, wave equation, a priori error estimate.

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1. Introduction

Electromagnetism is one application of the discontinuous Galerkin method, among many other areas. In [1], the discontinuous Galerkin method with solutions that are exactly divergence-free inside each element, is developped for numerically solving the Maxwell equations. In [2], M. Grote, A. Schneebeli and D. Schötzau propose and analyse the symmetric interior penalty discontinuous Galerkin method for the spatial discretization of Maxwell's equations in second order form.
Here, we consider a nonsymmetric interior penalty discontinuous Galerkin method to approximate in space an initial boundary value problem derived from Maxwell's equations in vacuum with perfect electric conductor boundary

$$\frac{\partial^2 u}{\partial t^2} + c^2 \nabla \times (\nabla \times u) = f, \qquad \nabla \cdot u = 0 \quad \text{in} \;\; \Omega \times I; \tag{1.1}$$

$$n \times u(x,t) = 0 \quad \text{on} \;\; \partial\Omega \times I, \;\; u(x,0) = u_0(x), \;\; \frac{\partial u}{\partial t}(x,0) = u_1(x) \quad \text{on} \;\; \Omega. \tag{1.2}$$

Here Ω is a convex polyhedron included in $\mathbb{R}^3$, $I = [0, t^*] \subset \mathbb{R}$, u_0 and u_1 are in $H_0(\nabla\times, \Omega) \cap H(\nabla\cdot 0, \Omega)$ and f is defined on $\Omega \times I$. Physically, u is the electric field, and f is related to a current density. Moreover $\mu_0 \varepsilon_0 c^2 = 1$ where $\mu_0 \approx 4\pi 10^{-7}$ H.m^{-1} and $\varepsilon_0 \approx (36\pi 10^9)^{-1}$F.m^{-1} are respectively the magnetic permeability and the electric permittivity in vacuum. If we assume that the domain Ω is "stable medium" with boundary and if u is the exact solution of (1.1) and (1.2) then u and $\nabla \times u$ belong to $H^1(\Omega)^3$. For the notations, if I is an interval, X is a function space and ϕ is a function on $\Omega \times I$ then $\|\phi\|_{L^p(I,X)}$ denotes the norm in $L^p(I)$ of the function $t \to \|\phi(\cdot, t)\|_X$. $L^p(X)$ is short for $L^p(I, X)$.
Let Π_h be a partition of Ω into tetrahedra and consider the same spaces and notations as in [5].
Faces: We define and characterise the faces of the triangulation Π_h. An interior face of Π_h is defined as the (non-empty) two-dimensional interior of $\partial K_1 \cap \partial K_2$, where K_1 and K_2 are two adjacents elements of Π_h. A boundary face of Π_h is defined as the (non-empty) two-dimensional interior of $\partial K \cap \partial\Omega$, where K is a boundary element of Π_h. We denote by F_h^I the union of all interior faces of Π_h, by F_h^D the union of all boundary faces of Π_h and by F_h the union of all faces of Π_h. Furthermore we identify F_h^D to $\partial\Omega$ since Ω is a polyhedron.

Traces: Let $H^s(\Pi_h) = \{v \ : \ v_{|K} \in H^s(K) \ \ \forall K \in \Pi_h\}$ for $s > \frac{1}{2}$ endowed with the norm $\|v\|^2_{s,\Pi_h} = \sum_{K\in\Pi_h} \|v\|^2_{s,K}$. Then the elementwise traces of functions in $H^s(\Pi_h)$ belongs to $TR(F_h) = \Pi_{K\in\Pi_h} L^2(\partial K)$; they are double-valued on F_h^I and single-valued on F_h^D. The space $L^2(F_h)$ can be identified with the functions in $TR(F_h)$ for which the two traces values coincide.
Traces operators: Let us introduce the following traces operators for piecewise smooth functions. First, let $w \in TR(F_h)^3$ and $e \subset F_h$. If e is an interior face in F_h^I, we denote by K_1 and K_2 the elements sharing e, by n_i the normal unit vector pointing exterior to K_i and we set $\omega_i = \omega_{|\partial K_i}, i = 1, 2$. We define the *average*, and the *tangential* and *normal jumps* of w at $x \in e$ as

$$\{\omega\} = \frac{\omega_1 + \omega_2}{2} \ , \quad [\omega]_T = n_1 \times \omega_1 + n_2 \times \omega_2 \ \text{ and } \ [\omega]_N = n_1 \cdot \omega_1 + n_2 \cdot \omega_2.$$

If $e \subset F_h^D$, we set for $x \in e$

$$\{\omega\} = \omega \ , \quad [\omega]_T = n \times \omega \ \text{ and } \ [\omega]_N = n \cdot \omega.$$

If $w \in H(\nabla\times, \Omega)$, then, for all $e \subset F_h^I$ the jump condition $[w]_T = 0$ holds true in $L^2(e)^3$ and if $w \in H(\nabla\cdot, \Omega)$, then, for all $e \subset F_h^I$ the jump condition $[w]_N = 0$ holds true in $L^2(e)$.
For $e \subset F_h$, we denote by $< \cdot, \cdot >_e$ the scalar product in $L^2(e)^3$ or $L^2(e)$, furthermore if F_h^D is identified to $\partial\Omega$, we identify $\sum_{e\subset F_h^D} < \cdot, \cdot >_e$ to $< \cdot, \cdot >$, the scalar product in $L^2(\partial\Omega)^3$ or $L^2(\partial\Omega)$.
Finite element spaces: In order to define the average of $\nabla \times u$, we set for $s > \frac{1}{2}$, $H^s(\nabla\times, \Pi_h) := \{v \ : \ v_{|K} \in H^s(K)^3$ and $\nabla \times (v_{|K}) \in H^s(K)^3, \forall K \in \Pi_h\}$. Let $p = (p_K)_{K\in\Pi_h}$ be a degree vector that assigns to each element $K \in \Pi_h$ a polynomial approximation order $p_K \geq 1$. The generic $hp-$finite element space of piecewise polynomials is given by

$$S^p(\Pi_h) = \{u \in L^2(\Omega) : u_{|K} \in S^{p_K}(K) \qquad \forall K \in \Pi_h\}$$

where $S^{p_K}(K)$ is the space of real polynomials of degree at most p_K in K.
We also set $\Sigma_h := S^p(\Pi_h)^3$.
Now, fix a face $e \subset F_h$ and define the local parameters h, p by $h := min(h_K, h_{K'})$, $p = max(p_K, p_{K'})$ in the case of interior faces and $h := h_K$, $p = p_K$ in the case of boundary faces [4].

2. Discretization and a priori error estimate

In order to derive a weak formulation of (1)-(2), we note that formulas (1) in [5] implies that for any u with $\nabla \times u \in H(\nabla\times, \Omega)$

$$c^2(\nabla \times (\nabla \times u), v) = c^2(\nabla \times u, \nabla \times v) + a(u, v)$$

where

$$a(u, v) = c^2 < n \times (\nabla \times u), v > -c^2 \sum_{e\subset F_h^I} < [v]_T, \{\nabla \times u\} >_e .$$

Now, we introduce the penalty term via the form

$$J^\sigma(u, v) = \sum_{e\subset F_h^I} < \sigma[u]_N, [v]_N >_e + \sum_{e\subset F_h} < \sigma[u]_T, [v]_T >_e \qquad u, v \in H^1(\nabla\times, \Pi_h)$$

where $\sigma := \kappa p^2/h$ is a stabilization parameter and κ is a constant supposed ≥ 1. We also define

$$A(u, v) = c^2(\nabla \times u, \nabla \times v) + a(v, u) - a(u, v) + J(u, v),$$

$$B(u, v) = A(u, v) + J^\sigma(u, v)$$

and

$$J(u, v) = (\nabla \cdot u, \nabla \cdot v).$$

2.1. *Properties of the bilinear form*

Now, we introduce a norm associated with the bilinear form B and set for $u \in H^1(\nabla\times, \Pi_h)$

$$\|u\|_h^2 = \|u\|^2 + \|c(\nabla\times u)\|^2 + \|\nabla\cdot u\|^2 + \|\tfrac{1}{\sqrt{\sigma}}\{\nabla\times u\}\|_{0,F_h}^2 + \|\sqrt{\sigma}[u]_N\|_{0,F_h^I}^2 + \|\sqrt{\sigma}[u]_T\|_{0,F_h}^2.$$

We start by studying the continuity of the bilinear forms introduced above. We have :

Proposition 2.1. *$\forall v, u \in H^1(\nabla\times, \Pi_h)$ there exists a constant C independent of h and p such that*

$$|A(u,v)| \leq C\|u\|_h\|v\|_h \text{ and } |J^\sigma(u,v)| \leq C\|u\|_h\|v\|_h.$$

Proof. The proof is easily deduced from the definition of A, J^σ, $\|\cdot\|_h$ and the Cauchy-Schwarz inequality. □

In order to study the coercivity of the bilinear form B, we start by introducing the following inequality of Poincaré-Friedrichs type valid for $u \in H^1(\Pi_h)^3$.

Lemma 2.1. *Let $u \in H^1(\Pi_h)^3$. Then there exists C independent of h and p such that*

$$\|u\|^2 \leq C(\|c(\nabla\times u)\|^2 + \|\nabla\cdot u\|^2 + \sum_{e\subset F_h}\|\sqrt{\sigma}[u]_T\|_{0,e}^2 + \sum_{e\subset F_h^I}\|\sqrt{\sigma}[u]_N\|_{0,e}^2)$$

Proof. The proof follows immediately from Lemma 3.1 in [5] since $\kappa p^2 \geq 1$. □

Now, the following coercivity result holds.

Proposition 2.2. *There exists two constants $\alpha > 0$ and $\tilde{C} > 0$ independent of h and p such that*

$$B(v,v) \geq \alpha\|v\|_h^2 + \tilde{C}J^\sigma(v,v) \quad \forall v \in \Sigma_h.$$

Proof. Let us first recall the following inverse inequality

$$\|q\|_{0,\partial K}^2 \leq C_{inv}\frac{p_K^2}{h_K}\|q\|_{0,K}^2 \quad \forall q \in S^{p_K}(K). \tag{2.1}$$

with a constant $C_{inv} > 0$, only depending on the shape regularity of the mesh. Now, Let α be an arbitrary real number and choose $v \in \Sigma_h$. Then

$$B(v,v) - \alpha\|v\|_h^2 = (1-\alpha)(A(v,v) + J^\sigma(v,v)) - \alpha\int_{F_h}\{\nabla\times v\}^2/\sigma ds - \alpha\|v\|^2.$$

Since $\{\nabla\times v\}$ is the average of the flux at the face of two elements K_i and K_j, the corresponding integral can be split into two integrals with integrands $(\nabla\times v)_i/\sigma$ and $(\nabla\times v)_j/\sigma$, each one associated with the elements K_i or K_j respectively. Therefore, let $e \subset F_h$ and consider the integral associated with the element K. Using the inverse inequality (3), we have since $\nabla\times\ (\Sigma_h) \subset \Sigma_h$,

$$\int_e(\nabla\times v)^2/\sigma ds = \frac{1}{\sigma}\|\nabla\times v\|_{0,e}^2 \leq \frac{C_{inv}}{\sigma}\frac{p_K^2}{h_K}\|\nabla\times v\|_{0,K}^2 \tag{2.2}$$

so that, selecting σ to be equal to $\kappa p^2/h$ in (2.2), we obtain

$$-\int_e(\nabla\times v)^2/\sigma ds \geq -\frac{C_{inv}}{\kappa}\|\nabla\times v\|_{0,K}^2.$$

In particular,

$$-\int_{F_h}\{\nabla\times v\}^2/\sigma ds \geq -\frac{C_{inv}}{\kappa}\sum_{K\in\Pi_h}\|\nabla\times v\|_{0,K}^2 \geq -\frac{C_{inv}}{\kappa}A(v,v).$$

It then follows that

$$B(v,v)-\alpha\|v\|_h^2 \geq (1-\alpha-\alpha\frac{C_{inv}}{\kappa})A(v,v)+(1-\alpha)J^\sigma(v,v)-\alpha\|v\|^2$$

and we can easily see there exists a positive constant C such that

$$||u||^2 \leq C(A(v,v)+J^\sigma(v,v)).$$

Then, we obtain

$$\begin{aligned} B(v,v)-\alpha\|v\|_h^2 &\geq (1-\alpha C-\alpha-\alpha C_{inv}/\kappa)A(v,v)+(1-\alpha-\alpha C)J^\sigma(v,v) \\ &\geq (1-\alpha(C+1+C_{inv}/\kappa))A(v,v)+(1-\alpha(1+C))J^\sigma(v,v). \end{aligned}$$

Thus, if α is chosen for example $\alpha=\frac{1}{C+1+C_{inv}/\kappa}$ and $\tilde{C}=1-\frac{1+C}{C+1+C_{inv}/\kappa}>0$, we immediately obtain the coercivity result. □

Now, the following $hp-$approximation result to interpolate scalar function holds (see [4]).

Prop 2.1. Let $K\in\Pi_h$ and suppose that $u\in H^{t_K}(K),\ t_K\geq 1$. Then there exists a sequence of polynomials $\pi_{p_K}^{h_K}(u)\in S^{p_K}(K),\ p_K=1,2...$ satisfying, $\forall 0\leq q\leq t_K$

$$\|u-\pi_{p_K}^{h_K}(u)\|_{q,K}\leq C\frac{h_K^{min(p_K+1,t_K)-q}}{p_K^{t_K-q}}\|u\|_{t_K,K} \quad \text{and}$$

$$\|u-\pi_{p_K}^{h_K}(u)\|_{0,\partial K}\leq C\frac{h_K^{min(p_K+1,t_K)-\frac{1}{2}}}{p_K^{t_K-\frac{1}{2}}}\|u\|_{t_K,K}.$$

The constant C is independent of $u,\ h_K$ and p_K, but depends on the shape regularity of the mesh.

In order to interpolate vector function, we define

Definition 2.1. For $u=(u_1,u_2,u_3)$ we define
$\mathbf{\Pi}_p^h : H^t(\nabla\times,\Pi_h)\longrightarrow\Sigma_h$ *by* $\mathbf{\Pi}_p^h(u)=(\pi_p^h(u_1),\pi_p^h(u_2),\pi_p^h(u_3))$ with π_p^h is defined by $\pi_p^h(u)_{|K}=\pi_{p_K}^{h_K}(u_{|K})$ where $\pi_{p_K}^{h_K}$ is given by the previous Proposition.

2.2. *A priori error estimate*

The interior penalty finite element approximation to u is to find $U: I\longrightarrow\Sigma_h$ such that

$$(U_{tt},v)+B(U,v)=(f,v)\ \ \forall v\in\Sigma_h,\ \ U(0)=\mathbf{\Pi}_p^h(u_0),\ \ U_t(0)=\mathbf{\Pi}_p^h(u_1). \tag{2.3}$$

Upon choice of a basis for Σ_h and the data f, (2.3) determines U as the only solution to an initial value problem for a linear system of ordinary differential equations. Note that, if u is the exact solution of (1.1)-(1.2), then u satisfies the first equation in (2.3) and thus the problem is consistent.
We now analyse the proposed procedure by the method of energy estimates. In this Section, u denotes the exact solution of (1.1)-(1.2) and U the discrete solution of (2.3). C is generic constant independent of h and p which takes different values at the different places and depends on $\alpha,\ \tilde{C}$ the coercivity constants of the form B, t^* and Ω.
Let $\zeta=U-u$, then ζ satisfies

$$(\zeta_{tt},v)+B(\zeta,v)=0 \qquad \forall v\in\Sigma_h.$$

Decompose ζ as μ - ν where $\mu=\mathbf{\Pi}_p^h(u)-u$ and $\nu=\mathbf{\Pi}_p^h(u)-U$. Thus

$$(\nu_{tt},v)+B(\nu,v)=(\mu_{tt},v)+B(\mu,v) \qquad \forall v\in\Sigma_h.$$

Since $\nu_t(t) \in \Sigma_h$, we can set $v = \nu_t(t)$, obtaining

$$\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}\|\nu_t(t)\|^2 + \tfrac{1}{2}\tfrac{d}{dt}B(\nu(t),\nu(t)) &= (\mu_{tt}(t),\nu_t(t)) + B(\mu(t),\nu_t(t))\\ &\leq \tfrac{1}{2}\|\mu_{tt}(t)\|^2 + \tfrac{1}{2}\|\nu_t(t)\|^2 + B(\mu(t),\nu_t(t)).\end{aligned}$$

So

$$\frac{d}{dt}\|\nu_t(t)\|^2 + \frac{d}{dt}B(\nu(t),\nu(t)) \leq \|\mu_{tt}(t)\|^2 + \|\nu_t(t)\|^2 + 2B(\mu(t),\nu_t(t)).$$

Since $\nu_t(0) = \nu(0) = 0$, integration over $[0,t] \subset I$, yields

$$\|\nu_t(t)\|^2 + B(\nu(t),\nu(t)) \leq \|\mu_{tt}\|^2_{L^2(L^2)} + \int_0^t \|\nu_t(t)\|^2 dt + 2\int_0^t B(\mu(t),\nu_t(t))dt.$$

The final term may be integrated by parts in time. Hence,

$$2\int_0^t B(\mu(t),\nu_t(t))dt \leq 2|B(\mu(t),\nu(t))| + 2\int_0^t |B(\mu_t(t),\nu(t))|dt.$$

Therefore, we can apply the coercivity and continuity of B to get

$$\begin{aligned}&\|\nu_t(t)\|^2 + \alpha\|\nu(t)\|_h^2 + \tilde{C}J^\sigma(\nu(t),\nu(t))\\ &\leq \|\mu_{tt}\|^2_{L^2(L^2)} + \int_0^t \|\nu_t(t)\|^2 dt + C\|\nu(t)\|_h\|\mu(t)\|_h + 2\int_0^t |B(\mu_t(t),\nu(t))|dt\\ &\leq \|\mu_{tt}\|^2_{L^2(L^2)} + \int_0^t \|\nu_t(t)\|^2 dt + C\|\mu(t)\|_h^2 + \frac{\alpha}{2}\|\nu(t)\|_h^2 + C\int_0^t \left(\|\mu_t(t)\|_h^2 + \|\nu(t)\|_h^2\right) dt\\ &\leq C\left(\|\mu_{tt}\|^2_{L^2(L^2)} + \sup_{t\in I}\|\mu(t)\|_h^2 + \int_0^{t^*} \|\mu_t(t)\|_h^2 dt\right) + \frac{\alpha}{2}\|\nu(t)\|_h^2 + C\int_0^t \left(\|\nu_t(t)\|^2 + \|\nu(t)\|_h^2\right) dt.\end{aligned}$$

In particular,

$$\begin{aligned}&\|\nu_t(t)\|^2 + \|\nu(t)\|_h^2 + J^\sigma(\nu(t),\nu(t))\\ &\qquad\leq C\left(\|\mu_{tt}\|^2_{L^2(L^2)} + \sup_{t\in I}\|\mu(t)\|_h^2 + \int_0^{t^*} \|\mu_t(t)\|_h^2 dt\right) + C\int_0^t \left(\|\nu_t(t)\|^2 + \|\nu(t)\|_h^2\right)dt\end{aligned}$$

As this holds for all $t \in I$, Gronwall's Lemma implies that

$$\|\nu_t(t)\|^2 + \|\nu(t)\|_h^2 + J^\sigma(\nu(t),\nu(t)) \leq C\left(\|\mu_{tt}\|^2_{L^2(L^2)} + \sup_{t\in I}\|\mu(t)\|_h^2 + \int_0^{t^*} \|\mu_t(t)\|_h^2 dt\right).$$

Since $\zeta = \mu - \nu$ and $J^\sigma(\mu,\mu) = J^\sigma(\mu,\nu) = 0$,

$\|\zeta_t(t)\|^2 + \|\zeta(t)\|_h^2 + J^\sigma(\zeta(t),\zeta(t)) \leq C\left(\|\mu_{tt}\|^2_{L^2(L^2)} + \sup_{t\in I}\|\mu(t)\|_h^2 + \int_0^{t^*} \|\mu_t(t)\|_h^2 dt + \|\mu_t\|^2_{L^\infty(L^2)}\right)$. Thus, error bounds for the finite element approximation to the true solution reduces to the error bounds for the piecewise polynomial interpolant. Thus, we start by estimating $\|u - \mathbf{\Pi}_p^h(u)\|_h$ where $\mathbf{\Pi}_p^h$ is defined after definition 2.1. By using Proposition 2.1 and the definition of $\|\cdot\|_h$, we obtain the following estimates

$$\|u - \mathbf{\Pi}_p^h(u)\|_h^2 \leq C\sum_{K\in\Pi_h}\frac{h_K^{2\mu_K-2}}{p_K^{2t_K-3}}\|u\|^2_{t_K,K} \quad \text{and} \quad \|u - \pi_{p_K}^{h_K}(u)\|_{q,K} \leq C\frac{h_K^{\mu_K-q}}{p_K^{t_K-q}}\|u\|_{t_K,K} \quad \forall 0 \leq q \leq t_K.$$

where $\mu_K = min(p_K+1, t_K)$. By using the previous estimates, we can get the following result

Proposition 2.3. *Let $\mu_K = min(p_K+1, t_K)$ and u be the exact solution of (1.1)-(1.2). Suppose that $u_{|K} \in C^2(I, H^{t_K}(K)^3)$,$\forall K \in \Pi_h$ with $t_K \geq 2$. Let U the discrete solution of (2.3). Then, the error $\zeta = U - u$ satisfies*

$$\begin{aligned}&\|\zeta_t(t)\|^2 + \|\zeta(t)\|_h^2 + J^\sigma(\zeta(t),\zeta(t))\\ &\leq C\sum_{K\in\Pi_h}\frac{h_K^{2\mu_K-2}}{p_K^{2t_K-3}}\left(\|u_{tt}\|^2_{L^2(H^{t_K}(K)^3)} + \|u\|^2_{L^\infty(H^{t_K}(K)^3)} + \|u_t\|^2_{L^2(H^{t_K}(K)^3)} + \|u_t\|^2_{L^\infty(H^{t_K}(K)^3)}\right).\end{aligned}$$

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